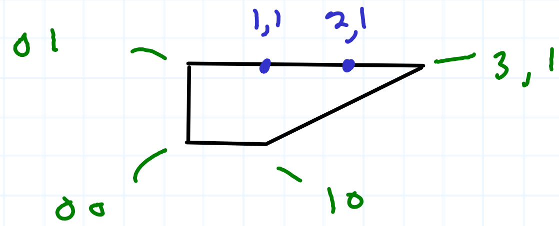
Example $H_2$, $E = D_3 + D_4$, $T = ((0,0), (1,0), (0,1), (1,1), (1,2), (3,1))$



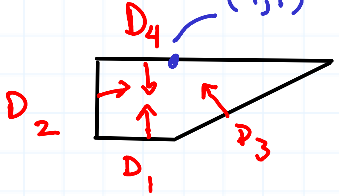

$P(E)$

Note: The exponents say how many steps are needed to slide each facet of the polytope to the corresponding lattice point

$$H_2 \longrightarrow \mathbb{P}^5$$

$$(x_1, x_2, x_3, x_4) \longrightarrow (\underset{u_1}{\underset{\parallel}{x_3x_4}}, \underset{u_2}{\underset{\parallel}{x_1x_4}}, \underset{u_3}{\underset{\parallel}{x_2x_3^3}}, \underset{u_4}{\underset{\parallel}{x_1x_2x_3^2}}, \underset{u_5}{\underset{\parallel}{x_1^2x_2x_3}}, \underset{u_6}{\underset{\parallel}{x_1^3x_2}})$$

$$\text{image}(H_2) = \{u \in \mathbb{P}^5 : u_2u_3 = u_1u_4,\ u_2u_4 = u_1u_5,\ u_2u_5 = u_1u_6,\ u_4^2 = u_3u_5,\ u_2u_4 = u_1u_5,\ u_2u_3 = u_1u_4)$$

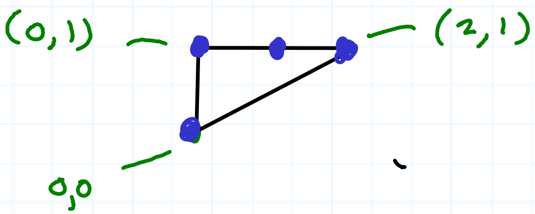


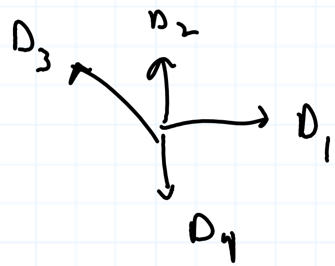

$(1,1) \longleftrightarrow x^{D_{(1,1)}+E} = x_1x_2x_3^2$

To get to $(1,1)$ requires sliding the $D_1$-facet 1 step, the $D_2$ facet 1 step, the $D_3$ facet 2 steps and the $D_4$ facets 0 steps: $x_1^1x_2^1x_3^2x_4^0 = x_1x_2x_3^2$.

**Example** $H_2$ , $E = D_4$

$$P(E) = \{ m \in \mathbb{R}^2 : m_1 \geq 0,\ m_2 \geq 0,\ -m_1 + 2m_2 \geq 0,\ -m_2 \geq -1 \}$$




Take $T = \{(0,0), (0,1), (1,1), (2,1)\}$

$$H_2 \longrightarrow \mathbb{P}^3$$

$$(x_1, x_2, x_3, x_4) \longmapsto (x_4,\ x_2 x_3^2,\ x_1 x_2 x_3,\ x_1^2 x_2)$$

$\sim (\lambda x_1, \mu x_2, \lambda x_3, \lambda^2 \mu x_4)$

On the open set where $x_3 \neq 0$ and $x_4 \neq 0$, this becomes

$$(x_1, x_2, 1, 1) \longmapsto (1,\ \underset{\substack{\| \\ a}}{x_2},\ \underset{\substack{\| \\ b}}{x_1 x_2},\ \underset{\substack{\| \\ c}}{x_1^2 x_2})$$

Singularity at $a = b = c = 0$

$b^2 = ac$

## Note

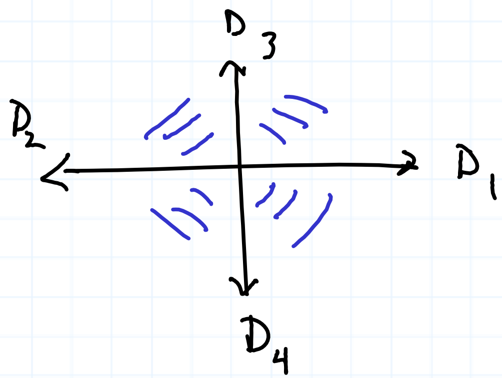
To get an embedding, you need to include the first lattice points in $P(E)$ next to each vertex, and the normal fan for $P(E)$ should be $\Delta$.

## Example

$$X = \mathbb{P}^1 \times \mathbb{P}^1$$

$$A^1 X = \bigoplus_{i=1}^{4} \mathbb{Z} D_i \Big/ (D_1 - D_2, D_3 - D_4) = \mathbb{Z}^2$$

$$\sum a_i D_i \longmapsto (a_1 + a_2, a_3 + a_4)$$




$$Z = \{ x \in \mathbb{C}^4 : x^{\hat{\sigma}} = 0 \quad \sigma \text{ max'l cone of } \Delta \}$$

$$= \{ x \in \mathbb{C}^4 : x_2 x_4 = x_1 x_4 = x_1 x_3 = x_2 x_3 = 0 \}$$

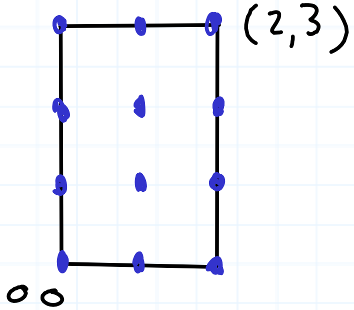
$= \{ x \in \mathbb{C}^4 : \ x_1 = x_2 = 0 \ \text{ or } \ x_3 = x_4 = 0 \}$

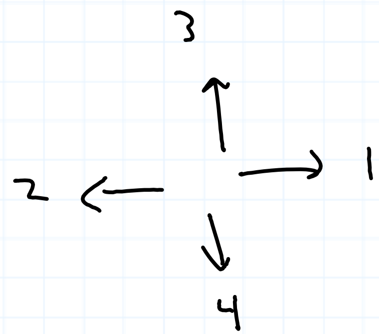
Group action: $G = \{ l \in (\mathbb{C}^*)^4 : \ l_1 l_2^{-1} = l_3 l_4^{-1} = 1 \}$

So

$$X = (\mathbb{C}^4 - Z) \big/_{(x_1, x_2, x_3, x_4) \sim (\lambda x_1, \lambda x_2, \mu x_3, \mu x_4)}$$

$P(2D_2 + 3D_4) =$




$\mathbb{P}^1 \times \mathbb{P}^1$

$$(x_1, x_2, x_3, x_4) \longrightarrow \Big( \underset{00}{x_2^2 x_4^3}, \ \underset{10}{x_1 x_2 x_4^3}, \ \underset{20}{x_1^2 x_4^3}, \ \underset{21}{x_2^2 x_3 x_4^2}, \ \underset{22}{x_1 x_2 x_3 x_4^2}, \ \underset{23}{x_1^2 x_3 x_4^2},$$

$$\dots, \ \underset{03}{x_2^2 x_3^3}, \ \underset{13}{x_1 x_2 x_3^3}, \ \underset{23}{x_1^2 x_3^3} \Big)$$

If $x_2 \neq 0$ and $x_4 \neq 0$, we can scale to get

$$(x_1, 1, x_3, 1) \to (1, x_1, x_1^2, x_3, x_1 x_3, x_1^2 x_3, x_3^2, x_1 x_3^2, x_1^2 x_3^2, x_3^3, x_1 x_3^3, x_1^2 x_3^3)$$

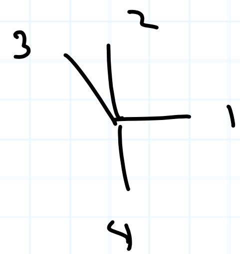
**Question:** What is the volume (i.e., surface area) of an embedded toric variety, $\nu : X(\Delta) \to \mathbb{P}^n$? restricted to just the real points. How does it depend on the divisor and chosen lattice points?

Thoughts: write $X(\Delta) \approx \mathbb{R}^\ell \setminus Z \big/ {x \sim \lambda x}$ and consider an open chart formed by setting appropriate coordinates equal to 1. We the have a mapping $\tilde{\nu} : \mathbb{R}^k \to U \subseteq S^n \subseteq \mathbb{R}^{n+1}$. Let $g = (\nu_{x_i} \cdot \nu_{x_j})_{1 \le i \le k,\ 1 \le j \le k}$

standard open chart of $\mathbb{P}^n$

and compute $\int_{\mathbb{C}^k} \sqrt{\det g}$.

**Example** Consider the mapping




$$\nu : H_2 \longrightarrow \mathbb{P}^5$$

$$(x_1, x_2, x_3, x_4) \longrightarrow (x_3 x_4,\ x_1 x_4,\ x_2 x_3^3,\ x_1 x_2 x_3^2,\ x_1^2 x_2 x_3,\ x_1^3 x_2)$$

discussed on the first page. Here $H_2 \cong \dfrac{\mathbb{C}^4 \setminus Z}{(x_1,x_2,x_3,x_4) \sim (\lambda x_1, \mu x_2, \lambda x_3, \lambda^2 \mu x_4)}$ $\lambda, \mu \in \mathbb{C}^*$

and $Z = \{ x_1 = x_3 = 0 \text{ or } x_2 = x_4 = 0 \}$.

On the open set $x_1 \neq 0$, $x_2 \neq 0$, each point in $H_2$ has a unique representative of the form $(1, 1, x_3, x_4)$ and $\nu$ restricted to this open set

$$(1, 1, x_3, x_4) \longrightarrow (x_3 x_4,\ x_4,\ x_3^3,\ x_3^2\ x_3,\ 1)$$

or

$$(x,y) \longmapsto \underset{\in \mathbb{P}^5}{(xy,\ y,\ x^3,\ x^2,\ x,\ 1)} \longrightarrow \frac{1}{r}(xy, y, x^3, x^2, x, 1) \in S^5 \subseteq \mathbb{R}^6$$

where $r = |(xy, y, x^3, x^2, x, 1)| = \sqrt{x^2y^2 + y^2 + x^6 + x^4 + x^2 + 1}$.