Math 411 Embeddings of toric varieties

X : smooth, complete toric variety

Δ : fan for $X = X(\Delta)$

$\Delta(1) = \{D_1, \ldots, D_\ell\}$: 1-dimensional cones of Δ

$S = \mathbb{C}[x_1, \ldots, x_\ell]$: homogeneous coordinate ring for X

If $E = \sum a_i D_i \in \mathbb{Z}^{\Delta(1)}$ (a divisor on X), we write $x^E = \prod_i x_i^{a_i} \in S$. The grading on S gives $\deg(x^E) = [E] \in A^1X$ where A^1X is given by the short exact sequence

$$0 \to M \to \mathbb{Z}^{\Delta(1)} \to A^1X \to 0$$

$$m \mapsto D_m := \sum_{i=1}^{\ell} \langle m, n_i \rangle D_i$$

where n_i is the first lattice point along D_i.

With the divisor $E = \sum a_i D_i$, we associate the polytope

$$P(E) = \{ m \in M_{\mathbb{R}} : \langle m, n_i \rangle \geq -a_i \text{ for } i=1,\dots,l \}$$

Let $T = \{m_0,\dots,m_t\} \subseteq M$ be a subset of the collection of lattice points in $P(E)$, including the vertices of $P(E)$.
In "homogeneous coordinates" for X, we get a mapping into projective space:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^t \\ x & \longmapsto & \left(x^{D_{m_0}+E}, \dots, x^{D_{m_t}+E}\right) \end{array}$$

where $D_m := \sum_{D\in\Delta(1)} \langle m, n_D\rangle D$ for $m \in M$. Since $[D_m] = 0 \in A^1X$, we have $[D_{m_i}+E] = [E]$ for all i. Hence, the mapping is "homogeneous of degree $[E]$".

Example $X = \mathbb{P}^n$ $\Delta(1) = \{D_0, D_1, \ldots, D_n\}$ where D_i is generated by $e_i \in \mathbb{Z}^n$ for $i = 1, \ldots, n$ and D_0 is generated by $-e_1 - \cdots - e_n$. The divisor class group is

$$A^1\mathbb{P}^n = \bigoplus \mathbb{Z}D_i \Big/ (D_1 - D_0, \ldots, D_n - D_0) \cong \mathbb{Z}$$

$$[\textstyle\sum a_i D_i] \longmapsto \textstyle\sum a_i$$

Let $E = dD_0$. Then

$$P(E) = \{ m \in \mathbb{R}^n : \langle m, -e_1 - \cdots - e_n \rangle \geq -d \text{ and } \langle m, e_i \rangle \geq 0 \ \forall i \}$$

$$= \{ m \in \mathbb{R}^n : \textstyle\sum m_i \leq d \text{ and } m_i \geq 0 \ \forall i \}$$

$$= \text{simplex spanned by } 0, de_1, \ldots, de_n,$$

Let $T = P(E) \cap \mathbb{Z}^n$ = all lattice points in $P(E)$.

Then $m \in T \Rightarrow D_m + E = \left[\sum_{j=0}^{n} \langle m, n_j \rangle D_j\right] + d D_0$

$$= (d - m_1 - \cdots - m_n) D_0 + m_1 D_1 + \cdots + m_n D_n ,$$

and thus $x^{D_m + E}$ has degree d. As m ranges over T, we get all monomials of degree d, giving rise to the so-called d-uple embedding of $\mathbb{P}^n$:

$$\mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d} - 1}$$

$$(x_0, \dots, x_n) \longmapsto (\underbrace{x_0^d, x_0^{d} x_1^{d-1}, \dots, x_n^d}_{})$$

← all monomials of degree d

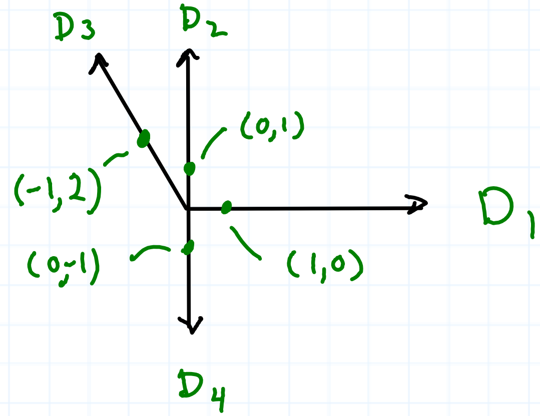
Sub-example: $n = d = 2$

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

$$(x, y, z) \longmapsto (x^2, xy, xz, y^2, yz, z^2)$$

These "corner" vertices are required in order for the mapping to be well-defined.

Example: $X = H_2$ (Hirzebruch surface, $a=2$)



$$A^1 H_a = \bigoplus \mathbb{Z} D_i \Big/ \langle D_1 - D_3,\ D_2 + 2D_3 - D_4 \rangle = \mathbb{Z}^2$$

$$\begin{aligned} D_1 &\longmapsto (1,0) \\ D_2 &\longmapsto (0,1) \\ D_3 &\longmapsto (1,0) \\ D_4 &\longmapsto (2,1) \end{aligned}$$

$0 = x_3 x_4 = x_1 x_4 = x_1 x_2 = x_2 x_3$

$x_1 = x_3 = 0$ or $x_2 = x_4 = 0$

$Z = \{ x^{\hat{\sigma}} = 0 \ \forall \text{ max'l cones } \sigma \}$

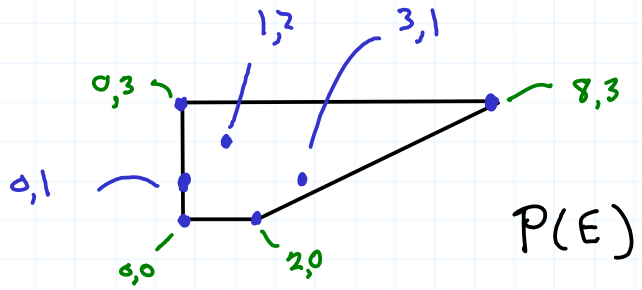
$= \{ x_3 x_4 = x_1 x_4 = x_1 x_2 = x_2 x_3 = 0 \}$

$= \{ x_1 = x_3 = 0 \text{ or } x_2 = x_4 = 0 \}$

$$H_2 = (\mathbb{C}^4 \setminus Z) \Big/ (x_1, x_2, x_3, x_4) \sim (\lambda x_1, \mu x_2, \lambda x_3, \lambda^2 \mu x_4)$$

Take $E = 2D_3 + 3D_4$. Then

$$P(E) = \{ m \in \mathbb{R}^2 : m_1 \geq 0,\ m_2 \geq 0,\ -m_1 + 2m_2 \geq -2,\ -m_4 \geq -3 \}.$$



Take $T = \{(0,0), (2,0), (0,1), (0,3), (1,2), (3,1), (8,3)\}$. Then

$$D_{00} + E = E = 2D_3 + 3D_4$$

$$D_{20} + E = 2D_1 - 2D_3 + 2D_3 + 3D_4 = 2D_1 + 3D_4$$

$$D_{01} + E = D_2 + 2D_3 - D_4 + 2D_3 + 3D_4 = D_2 + 4D_3 + 2D_4$$

etc.

$n_1 = 1,0$

$n_2 = 0,1$

$n_3 = -1,2$

$n_4 = 0,-1$

$$H_2 \longrightarrow \mathbb{P}^6$$

$$(x_1, x_2, x_3, x_4) \longmapsto (x_3^2 x_4^3,\ x_1^2 x_4^3,\ x_2^2 x_3^4 x_4^2,\ x_2^3 x_3^8 x_4^2,\ x_1 x_2^2 x_3^5 x_4,\ x_1^3 x_2 x_3 x_4^2,\ x_1^8 x_2^3)$$

$$(\lambda x_1, \mu x_2, \lambda x_3, \lambda^2 \mu x_4) \mapsto (\lambda^8 \mu^3 x_3^2 x_4^3,\ \lambda^8 \mu^3 x_1^2 x_4^3,\ \dots,\ \lambda^8 \mu^3 x_1^8 x_2^3)$$

$\underline{\underline{\text{equal as elements of } \mathbb{P}^6}}$