Math 411

Def. Let X be the toric variety corresponding to the fan Δ, and let $\Delta(1)$ be the one-dimensional rays of Δ. The homogeneous coordinate ring for X is

$$S = S_X := \mathbb{C}[x_D : D \in \Delta(1)]$$

s.e.s.:

$$0 \to M \to \mathbb{Z}^{\Delta(1)} \to A_{n-1}X \to 0$$

$$m \mapsto \sum_{D \in \Delta(1)} \langle m, n_D \rangle D$$

lattice

n_D = generator for D

where

$$\deg\Big(\prod_{D \in \Delta(1)} x_D^{a_D}\Big) := \sum a_D D \in A_{n-1}X \overset{\text{def}}{:=} \frac{\bigoplus \mathbb{Z} D_i}{\left(\sum_{D \in \Delta(1)} \langle m, n_D \rangle D : m \in M\right)}$$

Equals $A^1 X$ if X is smooth.

Example $\mathbb{P}^n$ The fan for $\mathbb{P}^n$ has rays $D_1, \dots, D_n$ generated by the standard basis vectors, and a ray D_{n+1} generated by $-e_1 - \cdots - e_n$. The homogeneous coordinate ring is $\mathbb{C}[x_1, \dots, x_{n+1}]$, graded by

$x_i := x_{D_i}$

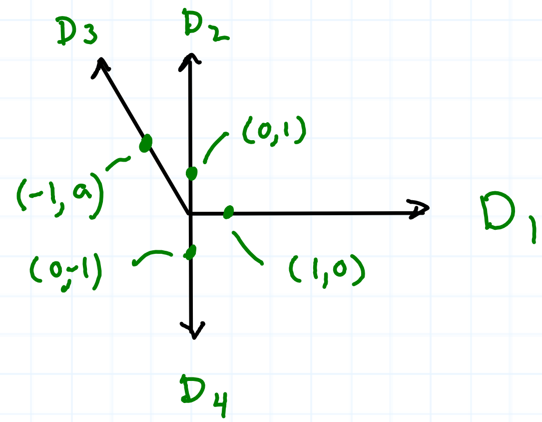
$$A^1\mathbb{P}^n = \bigoplus_{i=1}^{n+1} \mathbb{Z} D_i \Big/ (D_1 - D_{n+1}, \dots, D_n - D_{n+1}) \cong \mathbb{Z} D_{n+1} \cong \mathbb{Z}$$

$$D_i \longmapsto D_{n+1}$$

Thus, for example,

$$\deg\left(x_1 x_2^4 x_{n+1}^3\right) = D_1 + 4D_2 + 3D_{n+1} = 8D_{n+1} \in A^1\mathbb{P}^n$$

Example Hirzebruch surface H_a



$$S_{H_a} = \mathbb{C}[x_1, x_2, x_3, x_4]$$

$$x_i = D_i$$

graded by $A^1 H_a = \underbrace{\bigoplus \mathbb{Z} D_i}\big/\langle D_1 - D_3,\ D_2 + aD_3 - D_4 \rangle = \mathbb{Z}^2$

$D_1 = D_3$

$D_4 = D_2 + aD_3 = D_2 + aD_1$

$$D_1 \longmapsto (1,0)$$
$$D_2 \longmapsto (0,1)$$
$$D_3 \longmapsto (1,0)$$
$$D_4 \longmapsto (a,1)$$

For instance, $\deg\left(x_1 x_2^2 x_3^3 x_4^4\right) = (4+4a, 6)$. In general, $\deg\left(\prod x_i^{c_i}\right) = (c_1 + c_3 + a_4 c_4,\ c_2 + c_4)$.

Quotients and homogeneous coordinates

Let X be a simplicial toric variety, i.e., if each cone is generated by linearly independent generators (this includes all smooth toric varieties).

Let Δ be the fan for X. For each $\sigma \in \Delta$, define

$$x^{\hat{\sigma}} = \prod_{D \notin \sigma(1)} x_D$$

where $\sigma(1) = \Delta(1) \cap \sigma$ = one-dimensional cones of σ. Let

$$Z = \{ x \in \mathbb{C}^{\Delta(1)} : x^{\hat{\sigma}} = 0 \ \forall \sigma \in \Delta \}$$

It suffices to only use the maximal cones

Thm. (David Cox)

$$X = \mathbb{C}^{\Delta(1)} \setminus Z \,\Big/$$

group action to be described

Group action

$$A_{n-1}X := \mathbb{Z}^{\Delta(1)} \Big/ \Big\{ \sum_{D\in\Delta(1)} \langle m, n_D\rangle D : m \in M \Big\}$$

$$\begin{array}{ccccc} \mathbb{Z}^{\Delta(1)} & \longrightarrow & A_{n-1}X & \longrightarrow & 0 \\ D & \longmapsto & [D] & & \end{array}$$

$$\Big\Downarrow \ \hom(\cdot, \mathbb{C}^*)$$

$$\hom(A_{n-1}X, \mathbb{C}^*) \longrightarrow \hom(\mathbb{Z}^{\Delta(1)}, \mathbb{C}^*)$$

$$\hom(A_{n-1}X, \mathbb{C}^*) = G \cong \Big\{ \ell \in (\mathbb{C}^*)^{\Delta(1)} : \prod_{D\in\Delta(1)} \ell_D^{\langle m, n_D\rangle} = 1 : m \in M \Big\} \leq (\mathbb{C}^*)^{\Delta(1)} \subset \mathbb{C}^{\Delta(1)}$$

$$\hom(\mathbb{Z}^{\Delta(1)}, \mathbb{C}^*) \cong (\mathbb{C}^*)^{\Delta(1)}$$

Action of $\ell \in G$ on $x \in \mathbb{C}^{\Delta(1)}$: $\ell \cdot x := (\ell_D x_D)_{D\in\Delta(1)}$

Example $X = \mathbb{P}^n$, $\Delta(1) = \{D_0, \dots, D_n\}$,

$$A_{n-1}X = A^1X = \mathbb{Z}^{\Delta(1)} / \{D_0 - D_n, D_1 - D_n, \dots, D_{n-1} - D_n\} \cong \mathbb{Z}D_0 \cong \mathbb{Z}$$

$$\sum_{D \in \Delta(1)}{}' a_D D \longmapsto \sum{}' a_D$$

$$G = \{ l \in (\mathbb{C}^*)^{\Delta(1)} : l_{D_0} l_{D_n}^{-1} = 1, \dots, l_{D_{n-1}} l_{D_n}^{-1} = 1 \} = \{ l \in (\mathbb{C}^*)^{\Delta(1)} : l_0 = \cdots = l_n \in \mathbb{C}^* \}$$

$$Z = \{ x \in \mathbb{C}^{\Delta(1)} : x^{\hat{\sigma}} = 0 \ \underbrace{\forall \text{ max'l cones } \sigma \text{ for } \Delta}_{\substack{\text{each collection of } n \text{ cones from } \Delta(1) \\ \text{gives a max'l cone}}} \} = \{ x \in \mathbb{C}^{\Delta(1)} : x_{D_0} = \cdots = x_{D_n} = 0 \} = \{\vec{0}\}$$

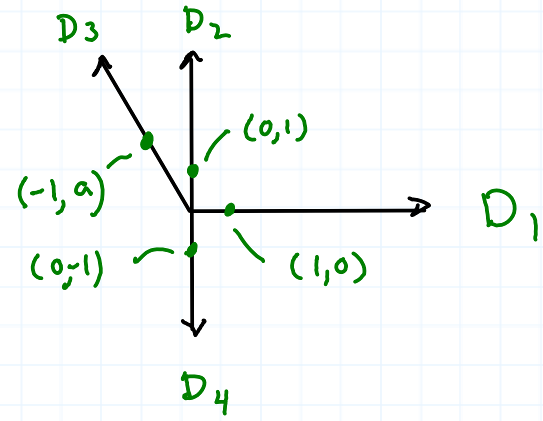
Thus, according to Cox's thm.,

$$\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{\vec{0}\} \Big/ x \sim lx$$

Example Hirzebruch, H_a

$$A_{n-1} H_a = A^1 H_a = \bigoplus \mathbb{Z} D_i \big/ \langle D_1 - D_3,\ D_2 + a D_3 - D_4 \rangle = \mathbb{Z}^2$$

$$G = \{ l \in \mathbb{C}^{\Delta(1)} : l_{D_3} = l_{D_1},\ l_{D_4} = l_{D_1}^a l_{D_2} \}$$



$$Z = \{ x \in \mathbb{C}^{\Delta(1)} : x^{\hat{\sigma}} = 0 \ \ \forall \text{ max'l } \sigma \}$$

$$= \{ x \in \mathbb{C}^{\Delta(1)} : x_{D_3} x_{D_4} = 0,\ x_{D_1} x_{D_4} = 0,\ x_{D_1} x_{D_2} = 0,\ x_{D_2} x_{D_3} = 0 \}$$

$$= \{ x \in \mathbb{C}^{\Delta(1)} : x_2 = x_4 = 0 \ \text{ or } \ x_1 = x_3 = 0 \}$$

So,

$$H_a = \left(\mathbb{C}^4 \setminus Z \right) \big/ (x_1, x_2, x_3, x_4) \sim (\lambda x_1, \mu x_2, \lambda x_3, \lambda^a \mu x_4)$$