Math 411

Geometry vs combinatorics I

1. Compactness A TV is compact iff its fan covers $N_{\mathbb{R}}$.
2. Smoothness A TV is smooth iff each cone of its fan is generated by part of a $\mathbb{Z}$-basis for N.
3. Orientability A smooth complex TV is symplectic, hence orientable.

Cohomology

compact as a manifold

Let X be a smooth, complete toric variety corresponding to the fan Δ. Let $\Delta(1)$ denote the one-dimensional cones of Δ. For each $D \in \Delta(1)$, let n_D be the first lattice point along D.

The **Chow ring** for X is

$$A^*(X) := \mathbb{Z}[D : D \in \Delta(1)] / I + J$$

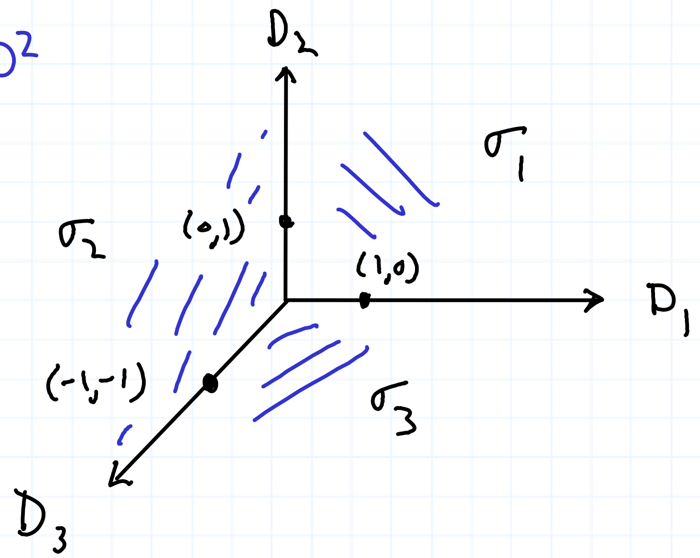
where each $D \in \Delta(1)$ is treated as an indeterminate and I and J are the ideals:

$$I = \Big(\prod_{D \in S} D : S \subseteq \Delta(1) \text{ s.t. } S \text{ does not span a cone of } \Delta \Big)$$

$$J = \Big(\sum_D \langle m, n_D \rangle D : m \in M \Big)$$

It suffices to let m run through a set of generators for M, say $e_1^*, \dots, e_n^*$.

Example: $\mathbb{P}^2$



$$\Delta(1) = \{D_1, D_2, D_3\}$$

$$I = (D_1 D_2 D_3)$$

$$J = \big(\langle e_i^*, (1,0) \rangle D_1 + \langle e_i^*, (0,1) \rangle D_2 + \langle e_i^*, (-1,-1) \rangle D_3 : i = 1, 2 \big)$$

Thus,

$$A^*\mathbb{P}^2 = \mathbb{Z}[D_1, D_2, D_3] / (D_1D_2D_3,\ D_1 - D_3,\ D_2 - D_3) \cong \mathbb{Z}[D_1]/(D_1^3)$$

$$D_1 \longleftarrow D_1$$

Thm. Let X be a smooth, complete toric variety. Then

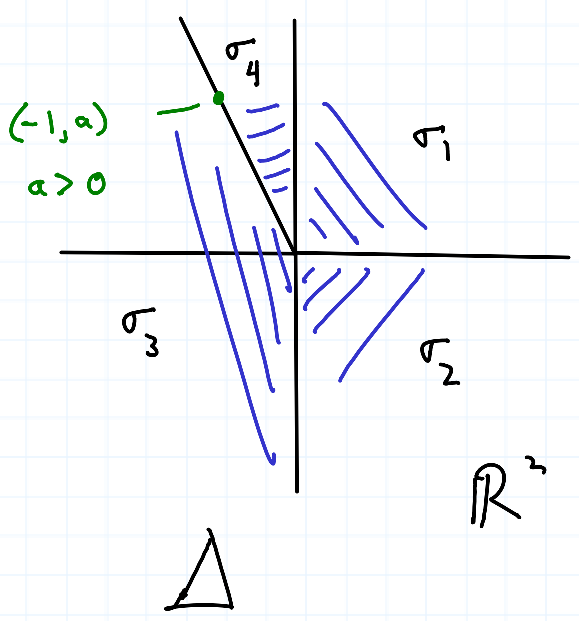
$$H^k X = \begin{cases} 0 & \text{for } k \text{ odd} \\ A^\ell X \otimes \mathbb{R} & \text{for } k = 2\ell \end{cases}$$

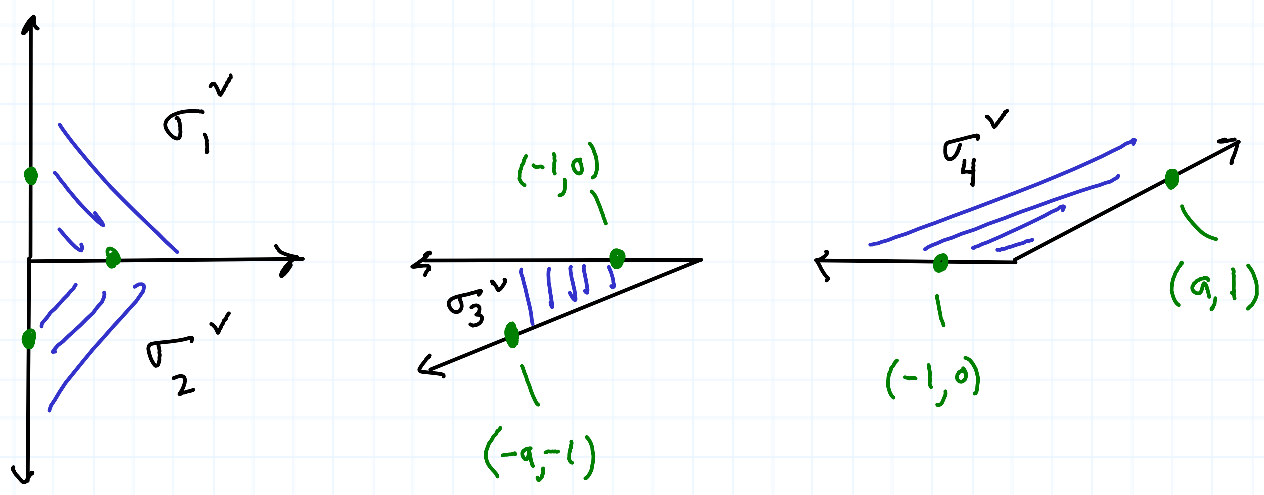
Example: $\mathbb{P}^2$

$$A^*\mathbb{P}^2 \cong \mathbb{Z}[D]/D^3 = \mathbb{Z} + \mathbb{Z}D + \mathbb{Z}D^2$$

k	0	1	2	3	4
$H^k\mathbb{P}^2$	$\mathbb{R}$	0	$\mathbb{R}$	0	$\mathbb{R}$

Hirzebruch Surfaces





Dual cones

$$\left.\begin{array}{llll} \mathbb{C}^2 \cong U_{\sigma_1} & (x,y) & (x^{-1}, x^{a} y) & U_{\sigma_4} \\ \mathbb{C}^2 \cong U_{\sigma_2} & (x,y^{-1}) & (x^{-1}, x^{-a} y^{-1}) & U_{\sigma_3} \end{array}\right] =: H_a = X(\Delta)$$

$$\downarrow \pi := \pi_1$$ (project to first coordinate)

$$U_1 \subseteq \mathbb{P}^1 \supseteq U_2 \qquad x \longmapsto 1/x$$

$\longleftarrow$ standard covering of $\mathbb{P}^1$.

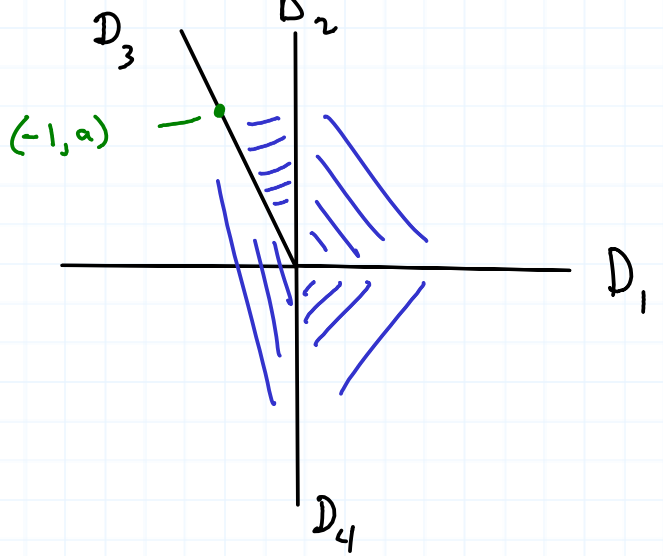
U_{σ_1} and U_{σ_2} glue together to get $\mathbb{C} \times \mathbb{P}^1$. So do U_{σ_3} and U_{σ_4}.

We have the surjective mapping $H_a \xrightarrow{\pi} \mathbb{P}^1$ such that if U is one of the standard chart domains:

$$\begin{tikzcd} \pi^{-1}U \arrow[r, phantom, "\simeq"] \arrow[d, "\pi"'] & \mathbb{C} \times \mathbb{P}^1 \arrow[d, "\pi_1"] \\ U \arrow[r, phantom, "\approx"] & \mathbb{C} \end{tikzcd}$$

In particular, $\pi^{-1}(\text{point}) \simeq \mathbb{P}^1$. In this sense, H_a is a $\mathbb{P}^1$-bundle.

Cohomology:



$$I = (D_1 D_3, D_2 D_4)$$

$$J = (D_1 - D_3, D_2 + a D_3 - D_4)$$

Note: $D_1 D_2 D_3$, etc., are already in this ideal.

$$A^* H_a = \mathbb{Z}[D_1, D_2, D_3, D_4] / (D_1 D_3, D_2 D_4, D_1 - D_3, D_4 + a D_3 - D_2)$$

$$\cong \mathbb{Z}[D_1, D_2] / (D_1^2, D_2(D_2 + a D_1))$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}\langle D_1, D_2 \rangle \oplus \mathbb{Z}\langle D_1 D_2 \rangle$$

k	0	1	2	3	4
$H^k(H_a)$	$\mathbb{R}$	0	$\mathbb{R}^2$	0	$\mathbb{R}$

(Note symmetry, guaranteed by Poincaré duality.)