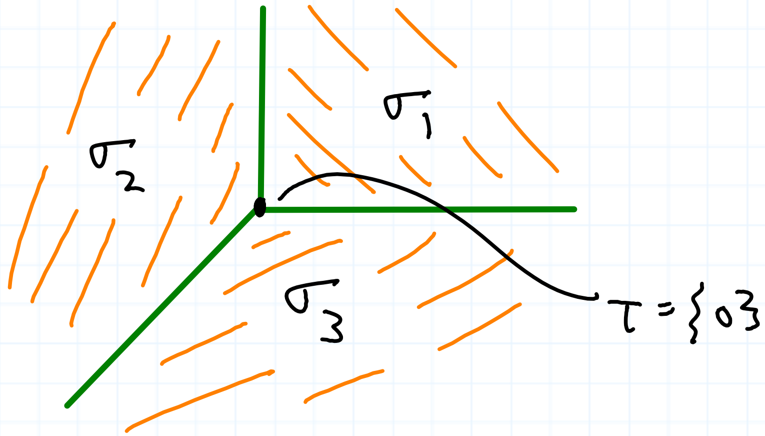
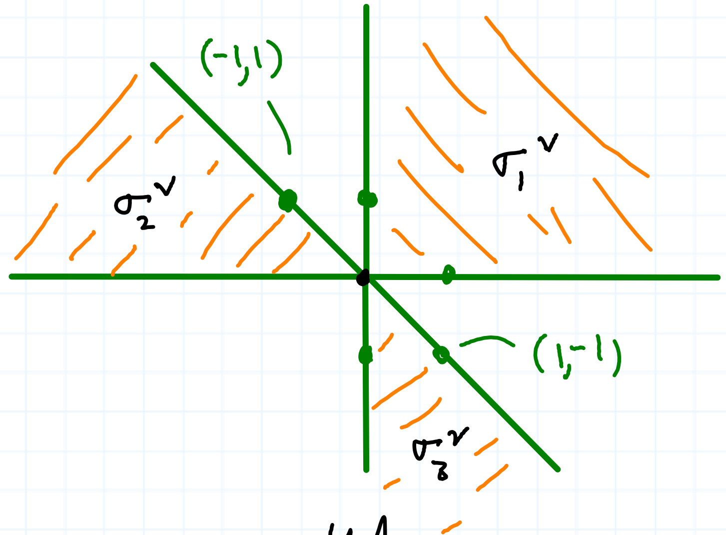
Math 411

Example $\mathbb{P}^2$

$\mathbb{R}^2$ (not $\mathbb{R}^3$)



$N = \mathbb{Z}^2$



M

$\tau^\vee = \mathbb{R}^2$

$u := e_{(1,0)}$

$v := e_{(0,1)}$

$$\begin{array}{lcl}
\mathbb{C}[S_{\sigma_1}] \cong \mathbb{C}[u,v] & \searrow & \\
\mathbb{C}[S_{\sigma_2}] \cong \mathbb{C}[\tfrac{v}{u}, \tfrac{1}{u}] & \longleftrightarrow & \mathbb{C}[S_\tau] \cong \mathbb{C}[u, \tfrac{1}{u}, v, \tfrac{1}{v}] \\
\mathbb{C}[S_{\sigma_3}] \cong \mathbb{C}[\tfrac{u}{v}, \tfrac{1}{v}] & \nearrow &
\end{array}$$

U_{σ_1}, U_{σ_2}, U_{σ_3} are copies of $\mathbb{C}^2$, $U_\tau \approx (\mathbb{C}^*)^2$

Algebraic group theorist's T^2 (2-torus)

Recall $\mathbb{P}^2$ as a manifold:

$$
\begin{aligned}
& U_{z \neq 0} \xrightarrow{\ \cong\ } \mathbb{C}^2 \\
& (x,y,z) \longmapsto \left(\tfrac{x}{z}, \tfrac{y}{z}\right) \\
\mathbb{P}^2 = \ & U_{y \neq 0} \xrightarrow{\ \cong\ } \mathbb{C}^2 \\
& (x,y,z) \longmapsto \left(\tfrac{x}{y}, \tfrac{z}{y}\right) \\
& U_{x \neq 0} \xrightarrow{\ \cong\ } \mathbb{C}^2 \\
& (x,y,z) \longmapsto \left(\tfrac{y}{x}, \tfrac{z}{x}\right)
\end{aligned}
$$

$$(u,v) \longmapsto \left(\tfrac{u}{v}, \tfrac{1}{v}\right)$$

$$(u,v) \longmapsto \left(\tfrac{v}{u}, \tfrac{1}{u}\right)$$

Torus (T^2) action on $\mathbb{P}^2$

$$U_\tau \hookrightarrow U_{\sigma_i}$$

In general, $U_{\{0\}} = T^n := (\mathbb{C}^*)^n$ acts on each U_σ via componentwise multiplication mediated by the inclusion $U_{\{0\}} \hookrightarrow U_\sigma$

Example

$$U_\tau = (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}^2 \cong U_{\sigma_2}$$

$$(u, v) \longmapsto \left(\frac{v}{u}, \frac{1}{u}\right)$$

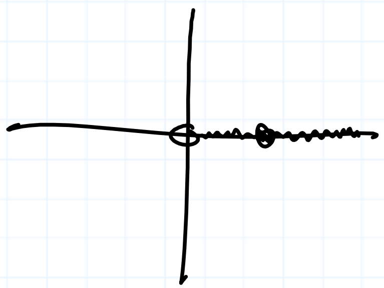
action: If $(u,v) \in U_\tau$ and $(p,q) \in U_{\sigma_2}$, then

$$(u,v) \cdot (p,q) = \left(\frac{v}{u} \cdot p, \frac{1}{u} \cdot q\right)$$

☆ Note: In general a toric variety has T^n as a dense subset

Orbits of this group action:

1. $\{(0,0)\}$
2. $\{(x,0) : x \in \mathbb{C}^*\}$
3. $\{(0,y) : y \in \mathbb{C}^*\}$
4. $\{(x,y) : x,y \in \mathbb{C}^*\}$



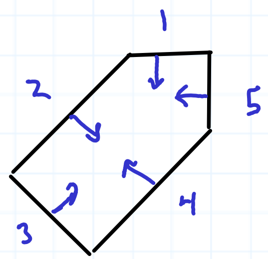
Toric varieties from polytopes

Let P be a polytope in $M_{\mathbb{R}}$.

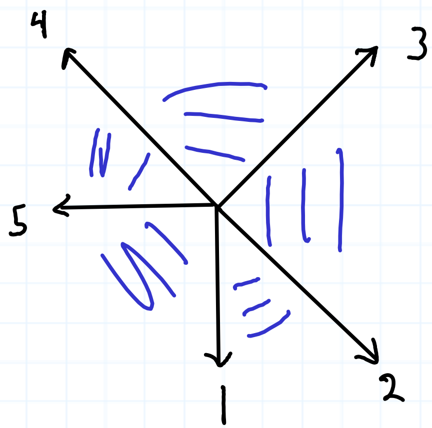
"Polytope" = convex hull of a finite set of points
= bounded intersection of a finite number of half-planes.

Let $\Delta(P)$ be the corresponding fan in $N_{\mathbb{R}}$ constructed from the inward-pointing normals to the faces of P

Note: The dual cones come from the angles at the vertices.

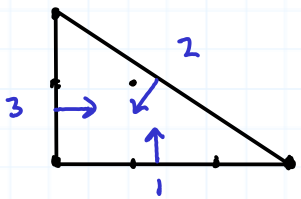


$P \subseteq M_{\mathbb{R}}$

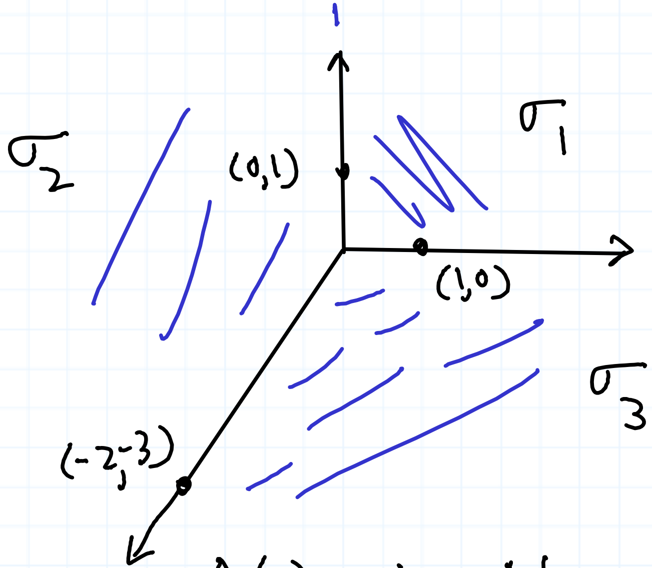


$\Delta(P)$ in $N_{\mathbb{R}}$

Example

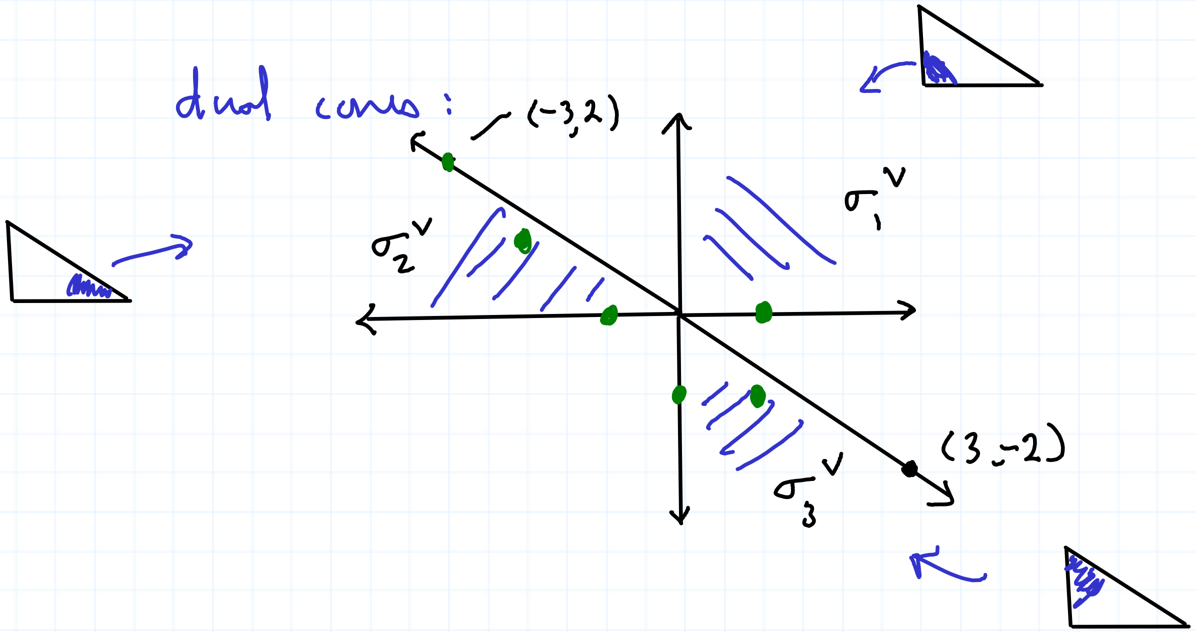


$P \subseteq M_{\mathbb{R}} = \mathbb{R}^2$



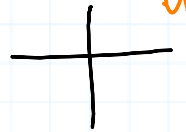
$\Delta(P)$ in $N_{\mathbb{R}} = \mathbb{R}^2$

dual cones:

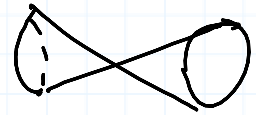


$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[x,y]$ $\qquad U_1 = \mathbb{C}^2$

$U_1 \cap \mathbb{R}^2$



$\mathbb{C}[S_{\sigma_2}] = \mathbb{C}\left[\frac{1}{x}, \frac{y}{x^2}, \frac{y^2}{x^3}\right] \simeq \mathbb{C}[u,v,w]/(uw-v^2)$ $\qquad U_2 = \{(u,v,w) \in \mathbb{C}^3 : v^2 = uw\}$



$U_2 \cap \mathbb{R}^3$

$\mathbb{C}[S_{\sigma_3}] = \mathbb{C}\left[\frac{1}{y}, \frac{x}{y}, \frac{x^3}{y^2}\right] \simeq \mathbb{C}[a,b,c]/(ac-b^3)$ $\qquad U_3 = \{(a,b,c) \in \mathbb{C}^3 : b^3 = ac\}$

$\approx U_3 \cap \mathbb{R}^3$

$$U_1 \longrightarrow U_2, \qquad (x,y) \longmapsto \left(\frac{1}{x}, \frac{y}{x^2}, \frac{y^2}{x^3}\right)$$

$$U_1 \longrightarrow U_3, \qquad (x,y) \longmapsto \left(\frac{1}{y}, \frac{x}{y}, \frac{x^3}{y^2}\right)$$

$$U_2 \longrightarrow U_3, \qquad (u,v,w) \longmapsto \left(\frac{uv}{w}, \frac{u}{v}, \frac{1}{w}\right)$$