Math 411

Def. A semi-Riemannian manifold is a manifold M with a scalar product $\langle\,,\rangle_p$ on T_pM $\forall p \in M$, varying smoothly with p. If the scalar product has index 0, then M is Riemannian.

Consider the entries in the matrix defining $\langle\,,\rangle$ locally.

We saw that a scalar product on a vector space V corresponds with an element of the symmetric product S^2V^*. Thus, we can define a semi-Riemannian manifold as a manifold with a section of the vector bundle S^2T^*M:

$$\begin{tikzcd} S^2T^*M \arrow[d, shift right, "\pi"'] \\ M \arrow[u, shift right, dashed, "g"'] \end{tikzcd}$$

such that the corresponding bilinear form on T_pM is nondegenerate $\forall p$.

Let M be an oriented semi-Riemannian n-manifold. For each $p \in M$, we get a volume form $\omega_{T_pM} \in \Omega^n M$ an a $*$-operator: $* : \Lambda^k T_p^* M \longrightarrow \Lambda^{n-k} T_p^* M$ varying smoothly with p. These glue together to give a volume form $\omega_V \in \Omega^n M$ and a $*$-operator

$$* : \Omega^k M \to \Omega^{n-k} M.$$

We get an induced coderivative mapping:

$$\begin{tikzcd} \Omega^k M \arrow[r, "d"] \arrow[d, "*"'] & \Omega^{k+1} M \arrow[d, "*"] \\ \Omega^{n-k} M \arrow[r, "(-1)^k \delta"] & \Omega^{n-k-1} M \end{tikzcd}$$

Lemma $** = (-1)^{n(n-k)+s}\, \mathrm{Id}$.

Hence, $*$ is an isomorphism.

Pf/ HW. □

Def. The codervivative $\delta_k : \Omega^{n-k} M \to \Omega^{n-k-1} M$ is defined by

$$\delta_k = (-1)^k * \circ d \circ *^{-1}$$

Goal We would like to use $*$ to get an isomorphism $H^k M \to H^{n-k} M$ for closed Riemannian manifolds.

Recall $H^k M = \ker(\Omega^k \xrightarrow{d} \Omega^{k+1} M) / \operatorname{im}(\Omega^{k-1} M \to \Omega^k M)$.

Let $\eta \in \Omega^k M$ with $d\eta = 0$, i.e., η is a cocycle.

Q. Is $*\eta$ a cocycle?

There is kind of commutation of $*$ and d:

Since $** = \pm id$

$$d*\eta = \pm ** d * \underbrace{**^{-1}}_{= id} \eta = \pm ** d *^{-1} \eta = \pm * \delta \eta$$

So $d * \eta = \pm \delta \eta$ and, similarly, $\delta * \eta = \pm * d\eta$.

Thus, we are led to consider:

Def. The *harmonic* k-forms on M are

$$\mathcal{H}ar(M) = \{ \eta \in \Omega^k M : d\eta = 0 \text{ and } \delta\eta = 0 \}.$$

Thm. If M is a closed Riemannian manifold

$$\begin{aligned} \mathcal{H}ar^k M &\longrightarrow H^k M \\ \eta &\longmapsto [\eta] \end{aligned}$$

is an isomorphism.

Pf/ Punt. $\square$

Prop. $* : \mathcal{H}ar^k M \to \mathcal{H}ar^{n-k} M$ exists and is an isomorphism.

Pf/ Let $\eta \in \mathcal{H}ar^k M$. So $d\eta = 0$ and $\delta\eta = 0$. Then

$$d * \eta = \pm \delta d \eta = 0 \quad \text{and} \quad \delta * \eta = \pm * d\eta = 0.$$

Hence, $*\eta$ is harmonic. Hence, $* : \mathcal{H}ar^k M \to \mathcal{H}ar^{n-k} M$ makes sense. It is an isomorphism since $** = \pm \mathrm{id}$. $\square$

↙ see note, below

Thm. (Poincaré duality) If M is a closed Riemannian* manifold, then

$$* : H^k M \xrightarrow{\cong} H^{n-k} M.$$

Example: $H^0 S^n \cong \mathbb{R} \Rightarrow H^n S^n \cong \mathbb{R}$ (spanned by the volume form).

* **Note:** Every manifold can be given a Riemannian metric by using the charts and a partition of unity. So Poincaré duality holds for oriented closed manifolds.

Cor. If M is a closed, connected, oriented n-manifold, then

$$\begin{aligned} H^n M &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_M \omega \end{aligned}$$

is an isomorphism.

Pf/ $H^n M \cong H^0 M \cong \mathbb{R} \Rightarrow H^n M = \mathbb{R}[\omega_M]$, where ω_M is the volume form. Note that locally $\int \omega_M$ is $\int dx_1 \wedge \cdots \wedge dx_n := \int_A 1 = \mathrm{vol}(A)$.
So $\int_M \omega_M$ really does give the volume. In particular, $\int_M \omega_M \neq 0$. $\square$