

# Math 411

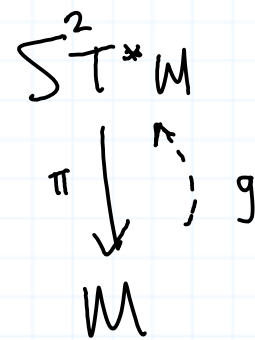
①

Def. A **semi-Riemannian** manifold is a manifold  $M$  with a scalar product  $\langle, \rangle_p$  on  $T_p M \forall p \in M$ , varying smoothly with  $p$ .  
If the scalar product has index 0, then  $M$  is **Riemannian**.

We saw that a scalar product on a vector space  $V$  corresponds with an element of the symmetric product  $S^2 V^*$ . Thus, we can define a

Consider the entries in the matrix defining  $\langle, \rangle$  locally.

semi-Riemannian manifold as a manifold with a section of the vector bundle  $S^2 T^* M$  such that the corresponding bilinear form on  $T_p M$  is nondegenerate  $\forall p$ .



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Let  $M$  be an oriented semi-Riemannian  $n$ -manifold.

For each  $p \in M$ , we get a volume form  $\omega_{T_p M} \in \Omega^n M$

and a  $*$ -operator:  $*$  :  $\Lambda^k T_p^* M \rightarrow \Lambda^{n-k} T_p^* M$

varying smoothly with  $p$ . These glue together to give

a **volume form**  $\omega_M \in \Omega^n M$  and a  **$*$ -operator**

$$* : \Omega^k M \rightarrow \Omega^{n-k} M.$$

We get an induced **codifferential mapping**:

$$\begin{array}{ccc}
 \Omega^k M & \xrightarrow{d} & \Omega^{k+1} M \\
 * \downarrow & & \downarrow * \\
 \Omega^{n-k} M & \xrightarrow{(-1)^k \delta} & \Omega^{n-k-1} M
 \end{array}$$

**Lemma**  $* * = (-1)^{n(n-k)+s} \text{Id}$ .

Hence,  $*$  is an isomorphism.

**Pf/ HW.**  $\square$

Def. The **codifferential**  $\delta_k: \Omega^{n-k}M \rightarrow \Omega^{n-k-1}M$  is defined by

$$\delta_k = (-1)^k * \circ d \circ *^{-1}$$

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**Goal** We would like to use  $*$  to get an isomorphism  $H^k M \rightarrow H^{n-k} M$  for closed Riemannian manifolds.

Recall  $H^k M = \ker(\Omega^k \xrightarrow{d} \Omega^{k+1} M) / \text{im}(\Omega^{k-1} M \rightarrow \Omega^k M)$ .

Let  $\eta \in \Omega^k M$  with  $d\eta = 0$ , i.e.,  $\eta$  is a cocycle.

Q. Is  $*\eta$  a cocycle?

There is kind of commutation of  $*$  and  $d$ :

Since  $** = \pm \text{id}$

$$d*\eta = \pm * \underbrace{d * *^{-1}}_{=\text{id}} \eta = \pm * * d *^{-1} \eta = \pm * \delta \eta$$

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So  $d^* \eta = \pm \delta \eta$  and, similarly,  $\delta^* \eta = \pm * d \eta$ .

Thus, we are led to consider:

Def. The **harmonic**  $k$ -forms on  $M$  are

$$\text{Har}(M) = \{ \eta \in \Omega^k M : d\eta = 0 \text{ and } \delta\eta = 0 \}.$$

Thm. If  $M$  is a closed Riemannian manifold

$$\text{Har}^k M \rightarrow H^k M$$

$$\eta \mapsto [\eta]$$

is an isomorphism.

Pf/ Punt.  $\square$

Prop.  $*$  :  $\text{Har}^k M \rightarrow \text{Har}^{n-k} M$  exists and is an isomorphism.

Pf/ Let  $\eta \in \text{Har}^k M$ . So  $d\eta = 0$  and  $\delta\eta = 0$ . Then

$$d*\eta = \pm \delta d\eta = 0 \quad \text{and} \quad \delta*\eta = \pm * d\eta = 0.$$

Hence,  $*\eta$  is harmonic. Hence,  $*$  :  $\text{Har}^k M \rightarrow \text{Har}^{n-k} M$  makes sense. It is an isomorphism since  $** = \pm \text{id}$ .  $\square$

Thm. (Poincaré duality) If  $M$  is a closed Riemannian <sup>★</sup> manifold, then

$$* : H^k M \xrightarrow{\cong} H^{n-k} M.$$

Example:  $H^0 S^n = \mathbb{R} \implies H^n S^n = \mathbb{R}$  (spanned by the volume form).

★ Note: Every manifold can be given a Riemannian metric by using the charts and a partition of unity. So Poincaré duality holds for oriented closed manifolds.

← see note, below

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Cor. If  $M$  is a closed, connected, oriented  $n$ -manifold, then

$$\begin{aligned} H^n M &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_M \omega \end{aligned}$$

is an isomorphism.

Pf/  $H^n M \cong H^0 M \cong \mathbb{R} \Rightarrow H^n M = \mathbb{R}[\omega_M]$ , where  $\omega_M$  is the volume form. Note that locally  $\int \omega_M$  is  $\int dx_1 \wedge \dots \wedge dx_n := \int_A 1 = \text{vol}(A)$ . So  $\int_M \omega_M$  really does give the volume. In particular,  $\int_M \omega_M \neq 0$ .  $\square$