Math 411

Warm-up. Given a nondegenerate bilinear form $\langle \, , \rangle$ on a finite-dimensional vector space V, last time we defined the "flat" isomorphism

$$\begin{aligned} V &\xrightarrow{\flat} V^* \\ v &\longmapsto \langle v, \cdot \rangle . \end{aligned}$$

How does this isomorphism compare to just choosing a basis for V and corresponding dual basis for V^*? For instance, say e_1, e_2, e_3 is an orthonormal basis for V with matrix $G = \mathrm{diag}(1,1,-1)$. Then

$$e_1^\flat(e_i) = \langle e_1, e_i \rangle = \delta_{1i} \Rightarrow e_1^\flat = e_1^* . \quad \text{Similarly, } e_2^\flat = e_2^* .$$

Finally, $e_3^\flat(e_i) = \langle e_3, e_i \rangle = -\delta_{3i} \Rightarrow e_3^\flat = -e_3^*$. The product on V^* induced by the flat-isomorphism is then given by for instance, $\langle e_3^*, e_3^* \rangle = \langle -e_3^\flat, -e_3^\flat \rangle = \langle e_3^\flat, e_3^\flat \rangle \overset{\text{induced by iso.}}{=} \langle e_3, e_3 \rangle$. Thus, the matrix for the product on V^* w.r.t. the dual basis is again G.

Let V be a finite-dimensional vector space / $\mathbb{R}$. Suppose V is oriented with $e_1, \ldots, e_n$ a positively-oriented, orthonormal basis.

Def. The volume form for V is $e_1^* \wedge \cdots \wedge e_n^*$

Prop. The volume form on V is the unique n-form sending each positively-oriented orthonormal basis to 1. More generally, let $v_1, \ldots, v_n$ be a positively-oriented basis for V, and let $G = (\langle v_i, v_j \rangle)$. Then

$$\omega_V = \sqrt{|\det G|} \; v_1^* \wedge \cdots \wedge v_n^* .$$

Thus, if $v_1, \ldots, v_n$ is, in addition, orthonormal, then $\omega_V = v_1^* \wedge \cdots \wedge v_n^*$.

Lemma Let $e_1, \dots, e_n$ and $v_1, \dots, v_n$ be ordered bases for V with corresponding dual bases $e_1^*, \dots, e_n^*$ and $v_1^*, \dots, v_n^*$ for V^*. Say $v_j = \sum_i a_{ij} e_i$ and $v_j^* = \sum_i b_{ij} e_i^*$. Define matrices $A = (a_{ij})$ and $B = (b_{ij})$. Then $B = (A^t)^{-1}$.

Pf/ HW. $\square$

Pf of Proposition/ Let $\tilde{I} = (\langle e_i, e_j \rangle)$ where $e_1, \dots, e_n$ is a positively-oriented orthonormal basis. Thus, $\tilde{I} = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n)$ where $\varepsilon_i = \{\pm 1\}$. Write $v_j = \sum_j a_{ij} e_i$ and let $A = (a_{ij})$. Then

$$G = A^t \, \tilde{I} \, A \qquad \text{(exercise)}$$

Taking determinants, $\det G = \det A^2 \implies \det A \overset{\star}{=} \sqrt{|\det G|}$ using the fact that $\det \tilde{I} = \pm 1$ and that $\det A > 0$ since $v_1, \dots, v_n$ is positively-oriented. Writing $v_j^* = \sum_i b_{ij} e_i^*$ and letting $B = (b_{ij})$, we have

$$v_1^* \wedge \cdots \wedge v_n^* = \det B \; e_1^* \wedge \cdots \wedge e_n^*$$

By the lemma, $\det B = (\det A)^{-1}$. Hence, by ☆,

$$e_1^* \wedge \cdots \wedge e_n^* = \sqrt{|\det G|} \; v_1^* \wedge \cdots \wedge v_n^* . \quad \square$$

Thm. For each $k \geq 0$, $\exists!$ linear mapping (the Hodge $*$-operator)

$$* : \Lambda^k V^* \longrightarrow \Lambda^{n-k} V^*$$

satisfying
$$\eta \wedge (* \zeta) = \langle \eta, \zeta \rangle \, \omega_V$$

$\forall \eta, \zeta \in \Lambda^k V^*$.

Pf/ Take $\mathcal{B} = \{e^*_\mu\}_{1\le\mu_1<\cdots<\mu_k\le n}$ as a basis for $\Lambda^k V^*$. The mapping $*$ is determined by its action on these basis vectors. So let $\zeta = e^*_\mu$ and write $*\zeta = \sum_\nu a_{\tilde\nu}\, e^*_{\tilde\nu}$. We would like to determine the coefficients, $\{a_{\tilde\nu}\}$. Let $\eta = e^*_\gamma \in \mathcal{B}$. Then

$$\langle \eta,\zeta\rangle = \langle e^*_\gamma,\zeta\rangle = \langle e^*_\gamma, e^*_\mu\rangle = \langle e_\gamma, e_\mu\rangle = \det(\langle e_{\gamma_i}, e_{\mu_j}\rangle) = \begin{cases} 0 & \text{if } \gamma\ne\mu \\ \varepsilon_\mu & \text{if } \gamma=\mu \end{cases}$$

where $\varepsilon_\mu = \prod_{i=1}^k \varepsilon_{\mu_i} \in \{\pm 1\}$. On the other hand,

$$e^*_\gamma \wedge *\zeta = a_{\tilde\gamma}\, e^*_\gamma \wedge e^*_{\tilde\gamma}$$

where $\tilde\gamma$ is the index formed by the complement of $\gamma_1,\dots,\gamma_k$ in $\{1,\dots,n\}$ arranged in increasing order. Letting τ_γ be the permutation sending $(1,\dots,n)$ to $(\gamma_1,\dots,\gamma_k,\tilde\gamma_1,\dots,\tilde\gamma_{n-k})$ we have

$$e^*_\gamma \wedge *\zeta = a_{\tilde\gamma}\,\mathrm{sgn}(\tau_\gamma)\,\omega_V.$$

The requirement $\eta \wedge *\zeta = \langle \eta, \zeta \rangle \, \omega_V$ for all η implies

$$a_{\tilde{\gamma}} \, \mathrm{sgn}(\tau_\gamma) = \langle e_\gamma^*, \zeta \rangle = \begin{cases} 0 & \text{if } \gamma \neq \mu \\ \varepsilon_\mu & \text{if } \gamma = \mu \end{cases}$$

Hence,

$$a_{\tilde{\gamma}} = \mathrm{sgn}(\tau_{\tilde{\gamma}}) \langle e^*_{\tilde{\gamma}}, \zeta \rangle = \begin{cases} 0 & \text{if } \gamma \neq \mu \\ \mathrm{sgn}(\tau_\mu)\, \varepsilon_\mu & \text{if } \gamma = \mu . \end{cases}$$

So

$$*\zeta := * e_\mu^* = a_{\tilde{\mu}} \, e^*_{\tilde{\mu}} = \mathrm{sgn}(\tau_\mu)\, \varepsilon_\mu \, e^*_{\tilde{\mu}}. \qquad \square$$

Summary

$$\boxed{* e_\mu^* = \mathrm{sgn}(\tau_\mu)\, \varepsilon_\mu \, e^*_{\tilde{\mu}}}$$

where (1) $\varepsilon_\mu = \prod_{i=1}^{k} \varepsilon_{\mu_i} = \pm 1$, (2) $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{n-k})$ where $\tilde{\mu}_1 < \cdots < \tilde{\mu}_h$ and $\{\tilde{\mu}_1, \ldots, \tilde{\mu}_{n-k}\} = \{1, 2, \ldots, n\} \setminus \{\mu_1, \ldots, \mu_k\}$, and (3) $\mathrm{sgn}(\tau_\mu)$ is the sign from "straightening out" $e_\mu \wedge e_{\tilde{\mu}}$. Hence, $e_\mu \wedge e_{\tilde{\mu}} = \mathrm{sgn}(\tau_\mu)\, \omega_V$.

$$= \mathrm{sgn}(\tau_\mu)\, e_1^* \wedge \cdots \wedge e_n^* .$$

Example Consider $\mathbb{R}^4$ with scalar product having orthonormal basis $e_1, \ldots, e_4$ and matrix $G = \operatorname{diag}\{1,1,1,-1\}$. Then

$*(e_1^* \wedge e_3^*) = \pm\, e_2^* \wedge e_4^*$ To find the sign, note that $(e_1^* \wedge e_3^*) \wedge (e_2^* \wedge e_4^*) = -\omega_V$

Therefore, $*(e_1^* \wedge e_3^*) = \varepsilon_1 \varepsilon_3 (-1)\; e_2^* \wedge e_4^* = -\, e_2^* \wedge e_4^*$

As another example, $*(e_1^* \wedge e_2^* \wedge e_4^*) = \operatorname{sgn}(\tau_{124})\, \varepsilon_1 \varepsilon_2 \varepsilon_4\, e_3^*$

$= (-1)[1 \cdot 1 \cdot (-1)]\, e_3^* = e_3^*$

Since $(e_1^* \wedge e_2^* \wedge e_4^*) \wedge e_3^* = -\omega_V$.

Example $\mathbb{R}^3$ with usual scalar product.

$*(e_1^* \wedge e_2^*) = e_3^*$, $*(e_1^* \wedge e_3^*) = -\, e_2^*$, $*(e_2^* \wedge e_3^*) = e_1^*$