Math 411

Thm. (Homotopy invariance) $f, g : M \to N$. If $f \sim g$, then

$$f^* = g^* : H^k N \to H^k M \quad \text{for all } k.$$

$h : [0,1] \times M \to N$

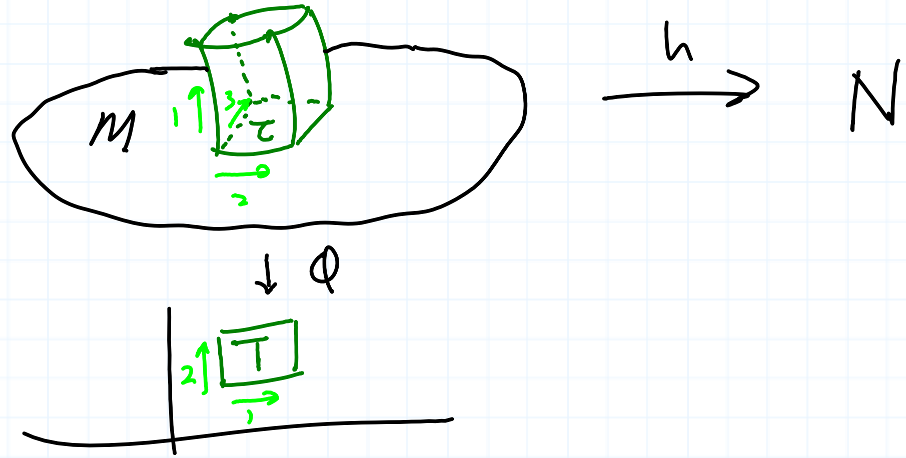
$h(0,x) = f(x)$, $h(1,x) = g(x)$

Problem: Given $\omega \in H^k N$, show $f^*\omega - g^*\omega = d\alpha$ for some $\alpha \in \Omega^{k-1} M$.

Motivation for proof: Take $\omega \in \Omega^k N$ with $d\omega = 0$. Fix some chart (U, ϕ) on M. Let T be a k-cell (= rectangle = product of k intervals) in $\phi(U) \subseteq \mathbb{R}^m$, with orientation induced by $\mathbb{R}^m$. Let $\tau = \phi^{-1}(T)$, a "k-cell" in M with orientation induced by T. Consider the prism, $[0,1] \times \tau \subseteq M$, with the product topology,

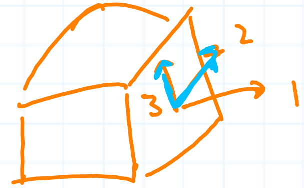
$\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$, t = coord. on $[0,1]$.

(skip orientation in lecture)

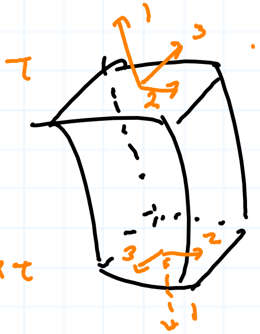


The boundary of $[0,1] \times T$ gets the usual boundary orientation.

Calculate: $\overset{\underset{\sim}{\partial}}{}$ $0 = \int_{[0,1]\times\tau} h^* \,d\omega = \int_{[0,1]\times\tau} d h^*\omega = \int_{\partial([0,1]\times\tau)} h^*\omega$



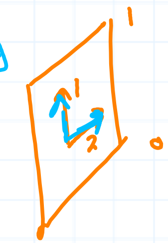
(2)



$$= \int_{top} h^*\omega + \int_{bottom} h^*\omega + \int_{sides} h^*\omega$$

The orientation on the sides coming from $\partial([0,1]\times\tau)$ is opposite that comming from the product orientation on $[0,1]\times\partial\tau$

$$= \int_\tau g^*\omega - \int_\tau f^*\omega - \int_{[0,1]\times\partial\tau} h^*\omega$$



$$\Longrightarrow \int_\tau g^*\omega - \int_\tau f^*\omega = \int_{[0,1]\times\partial\tau} h^*\omega$$

$$\overset{?}{=} \int_{\partial\tau}\left(\int_0^1 h^*\omega\right) = \int_{\partial\tau}\alpha = \int_\tau d\alpha.$$

we can hope

orientation of $\partial\tau$: $\boxed{\tau}$

This suggests finding α by integrating t out of $h^*\omega$.

Prism operator

$$P : \Omega^k([0,1]\times M) \longrightarrow \Omega^{k-1} M$$

$$\eta \longmapsto \underbrace{\int_0^1 \eta(\partial_t, -)}$$

omit this notation. Just use $P\eta$.

To define $\int_0^1 \eta(\partial_t, -)$, take an atlas $\{(U,\varphi)\}$ for M and corresponding atlas $\{([0,1]\times U, \mathrm{id}\times\varphi)\}$ for $[0,1]\times M$.

Suppose $\eta = \sum_\mu a_\mu(t,\vec{x})\, dt\wedge dx_\mu$ with respect to a chart $([0,1]\times U, \mathrm{id}\times\varphi)$

Then $P\eta = \sum_\mu \left[\int_0^1 a_\mu(t,\vec{x})\,dt\right] dx_\mu$. With respect to another chart $([0,1], \mathrm{id}\times\rho)$, say $\eta = \sum_\mu b_\mu(t,\vec{y})\, dt\wedge dy_\mu$ and $P\eta = \sum_\mu \left[\int_0^1 b_\mu(t,\vec{y})\,dt\right] dy_\mu$.

Let $\psi = \rho\circ\varphi^{-1}$, the transition function on M. Then $(\mathrm{id}\times\rho)\circ(\mathrm{id}\times\varphi)^{-1} = \mathrm{id}\times\psi$ and $(\mathrm{id}\times\psi)^*\left(\sum b_\mu\, dt\wedge dy_\mu\right) = \sum b_\mu\circ(\mathrm{id}\times\psi)\, dt\wedge d\psi_\mu = \sum a_\mu\, dt\wedge dx_\mu$.

Thus, $\sum\left[\int_0^1 a_\mu(t,\vec{x})\,dt\right] dx_\mu = \sum\left[\int_0^1 b_\mu(t,\psi(\vec{x}))\,dt\right] d\psi_\mu$

$$= \sum\left[\int_0^1 b_\mu(t,\psi(x))\,dt\right] d\psi_\mu = \psi^*\left(\sum \int_0^1 b_\mu(t,\vec{y})\,dt\right) dy_\mu.$$

So the local description glues together.

omit during lecture

If the local description for η with respect to $([0,1]\times U, \mathrm{id}\times\varphi)$ is $a(t,x)\,dx_\mu$, then $P\eta = 0$. Pulling back $a(t,x)\,dx_\mu$ along a transition function yields a differential form of the form $\sum b_\mu\, dy_\mu$, again not involving t. So again, everything glues together.

Example $M = \mathbb{R}^2$, $\eta = (xt+y^2)\,dx\wedge dy + (t^3y+x^2)\,dt\wedge dx$.

Then $P\eta = 0 + \left[\int_0^1 (t^3y+x^2)\,dt\right] dx = \left(\frac{1}{4}y + x^2\right) dx$

Theorem.* Let
$$i_0 : M \to [0,1]\times M \qquad \text{and} \qquad i_1 : M \to [0,1]\times M$$
$$x \mapsto (0,x) \qquad\qquad\qquad x \mapsto (1,x)$$

Let $\eta \in \Omega^{k-1}([0,1]\times M)$. Then $Pd\eta = i_1^*\eta - i_0^*\eta - dP\eta$.

Pf/ To come, below. $\square$

Pf. of homotopy invariance thm. / Let $\omega \in \Omega^k N$ with $d\omega = 0$; $f, g : M \to N$, and $f \overset{h}{\sim} g$. Then, using the notation in the theorem ☆ above,

$$g^*\omega - f^*\omega = (h \circ \iota_1)^*\omega - (h \circ \iota_2)^*\omega = \iota_1^* h^*\omega - \iota_2^* h^*\omega = Pd(h^*\omega) + dP(h^*\omega)$$

$$= P(h^* \underbrace{d\omega}_{0}) + dP(h^*\omega) \overset{☆☆}{=} dP(h^*\omega) \in d(\Omega^{k-1}M).$$

Hence $[g^*\omega] = [f^*\omega] \in H^k M$. $\square$

Cor.☆ Suppose M is contractible, i.e., id_M is homotopic to a constant mapping C via a homotopy h. Let $\omega \in \Omega^k M$, $k \geq 1$, with $d\omega = 0$. Then

$$\omega = d\left[P(h^*\omega)\right].$$

Pf/ $\omega = id_M^*\omega - \underbrace{C^*(\omega)}_{0} = dP(h^*\omega)$ by the proof of the homotopy invariance theorem (see ☆☆). $\square$

skip in lecture

(Pf. of thm. ✩ / The statement is linear in η and local in M.

There are two cases in local coordinates:

1. $\eta = a(t,\vec{x})\, dx_\mu$

Pf/ $d\eta = \dot{a}(t,\vec{x})\, dt \wedge dx_\mu + \sum_i \frac{\partial a}{\partial x_i} dx_i \wedge dx_\mu$

$\Longrightarrow P d\eta = \left[\int_0^1 \dot{a}(t,\vec{x})dt\right] dx_\mu = \left[a(1,\vec{x}) - a(0,x)\right] dx_\mu = i_1^* \eta - i_0^* \eta - \underbrace{dP\eta}_{0}$.

2. $\eta = a(t,x)\, dt \wedge dx_\mu$.

$i_1(t,\vec{x}) = (1,x) \Rightarrow i_{1,1} = 1 = \text{constant}$

$i_{0,1} = 0 = \text{constant}$

Pf/ In this case, $i_1^* \eta = 0$ and $i_0^* \eta = 0$ since $i_1^* dt = i_0^* dt = 0$.

We must therefore show that $P d\eta = -dP\eta$. Now,

$d\eta = \sum_i \frac{\partial a}{\partial x_i} dx_i \wedge dt \wedge dx_\mu = -\sum_i \frac{\partial a}{\partial x_i} dt \wedge dx_i \wedge dx_\mu \Longrightarrow$

$P d\eta = -\sum_i \left[\int_0^1 \frac{\partial a}{\partial x_i}(t,\vec{x})dt\right] dx_i \wedge dx_\mu$.

On the other hand,

$$P\eta = \left[\int_0^1 a(t,\vec{x})\,dt\right] \wedge dx_\mu \implies$$

$$dP\eta = \sum_i \left[\int_0^1 \frac{\partial a}{\partial x_i}(t,\vec{x})\,dt\right] \wedge dx_i \wedge dx_\mu . \quad \square$$

If $p \in X$, then $tp \in X$ for $t \in [0,1]$.

In lecture, just do $w = a(x)\,dx_1 \wedge \cdots \wedge dx_k$

Application Let $X \subseteq \mathbb{R}^n$, open, star-shaped with respect to 0.

Let $w \in \Omega^k \mathbb{R}^n$ with $dw = 0$. Say $w = \sum_\mu w_\mu \, dx_\mu$. We have $id_X \sim$ constant via $h: (t,x) = tx$. Therefore, by Cor. ☆, above, we have $w = d\alpha$ where

$$\alpha = Ph^* w = \sum_\mu P[w_\mu \circ h \; dh_\mu] = \sum_\mu P\left[w_\mu(tx)\left[x_{\mu_1} dt + t\,dx_{\mu_1}\right] \wedge \cdots \wedge \left[x_{\mu_k} dt + t\,dx_{\mu_k}\right]\right]$$

$$= \sum_\mu P\left[w_\mu(tx)\, t^k\, dx_\mu\right] + \sum_\mu P\left[w_\mu(tx) \sum_{i=1}^k (-1)^{i+1} t^{k-1} x_{\mu_i}\, dt \wedge dx_{\mu_1} \wedge \cdots \wedge \widehat{dx_{\mu_i}} \wedge \cdots \wedge dx_{\mu_k}\right]$$

$$= \sum_\mu \sum_{i=1}^k (-1)^{i+1} \left[\int_0^1 x_{\mu_i} w_\mu(tx)\, t^{k-1}\, dt\right] dx_{\mu_1} \wedge \cdots \wedge \widehat{dx_{\mu_i}} \wedge \cdots \wedge dx_{\mu_k}$$

Cor. Let F be a vector field on $\mathbb{R}^3$ such that $\operatorname{curl} F = 0$. Define $\varphi: \mathbb{R}^3 \to \mathbb{R}$ by $\varphi(p) = \int_\gamma F \cdot d\gamma$ where $\gamma(t) = tp$ for $t \in [0,1]$. Then $\nabla \varphi = F$. Pf/HW. $\square$