Math 411

$$0 \to \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \cdots$$

$$H^k M = \ker(\Omega^k M \xrightarrow{d} \Omega^{k+1} M) \big/ \operatorname{im}(\Omega^{k-1} M \xrightarrow{d} \Omega^k M)$$

$H^\bullet M = \bigoplus_{k \geq 0} H^k M$ is a graded, anti-commutative k-algebra under $\wedge$-product.

$M \to H^\bullet M$ is a contravariant functor (HW).

Last time: $H^\bullet \mathbb{R} = \mathbb{R} \oplus 0 \oplus 0 \oplus 0 \oplus \cdots$

$H^\bullet \mathbb{R}^3$ is related to grad, curl, div.

First Properties of $H^\bullet$

Prop. If $\dim M = n$, then $H^k M = 0$ for $k > n$

Pf/ $\Omega^k M = 0$ for $k > n$. $\square$

Prop. $H^0 M \approx \mathbb{R}^c$ where c is the number of connected components of M. (So $H^0 M \approx \mathbb{R}$ if M is connected.)

Pf/ HW. $\left(f \in \ker(\Omega^0 M \xrightarrow{d} \Omega^1 M) \Rightarrow df = \sum \frac{\partial f}{\partial x_i} dx_i = 0 \right.$

$\left. \Rightarrow \frac{\partial f}{\partial x_i} = 0 \ \forall i \Rightarrow f \text{ locally constant.} \right)$ $\square$

Prop. M orientable, closed (i.e., compact with $\partial M = \emptyset$), $\dim M = n$

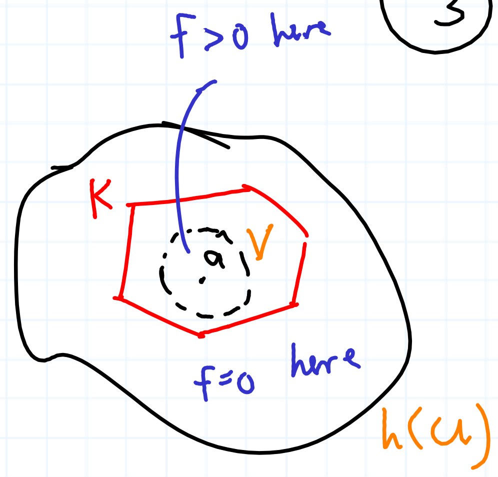
$\Rightarrow H^n M \neq 0$.

Pf/ $H^n M = \ker(\Omega^n M \xrightarrow{d} 0) / \operatorname{im}(\Omega^{n-1} M \xrightarrow{d} \Omega M) = \Omega^n M / d(\Omega^{n-1} M)$

By Stokes', $\omega \in \Omega^{n-1} M \Rightarrow \int_M d\omega = \int_{\partial M} \omega = \int_\emptyset \omega = 0$. Thus,

$\int_M \eta = 0 \ \forall \eta \in d(\Omega^{n-1} M)$. However, there are clearly elements

$\eta \in \Omega^n M$ such that $\int_M \eta \neq 0$. For instance, consider a bump function.



In detail, pick any chart (U, h) and a smooth function $f : h(U) \to \mathbb{R}$ such that $f(x) \geq 0 \;\forall\, x \in h(U)$, $f(x) = 0$ for $x \in h(U) \setminus K$ for some compact set $K \subseteq f(U)$ and $f(x) > 0$ for x is some open ball $V \subseteq K$. Define $\omega \in \Omega^n M$ by

$$\omega(p) = \begin{cases} f \circ h \; dx_1 \wedge \cdots \wedge dx_n & \text{for } p \in U \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int_M \omega > 0$.

Prop ☆. Suppose $f : M \to N$ is constant. Then $f^* H^k N \longrightarrow H^k M$ is the zero-mapping for $k > 0$. If M, N are connected, then $f^* : \underset{\mathbb{R}}{\underset{\wr\wr}{H^0 N}} \longrightarrow \underset{\mathbb{R}}{\underset{\wr\wr}{H^0 M}}$ is the identity mapping on $\mathbb{R}$. Pf/ HW. $\square$

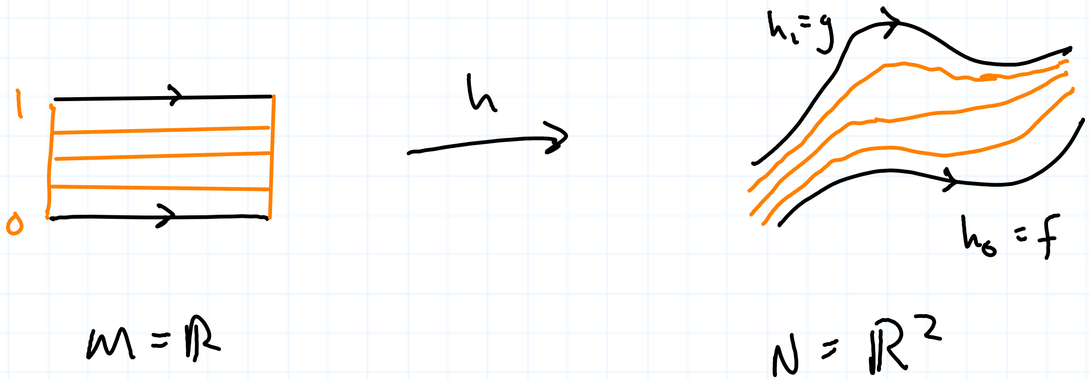
Homotopy invariance

Def. A homotopy between two mappings $f, g : M \to N$ is a mapping

$$h : [0,1] \times M \longrightarrow N$$

such that $h_0(x) := h(0,x) = f(x)$ and $h_1(x) := h(1,x) = g(x)$ $\forall x \in M$.

Picture



Notation: $f \sim g$ or $f \overset{h}{\sim} g$

Thm. (Homotopy invariance) Suppose $f, g : M \to N$ with f homotopic to g. Then $f^* = g^* : H^k N \to H^k M \quad \forall k \geq 0$.

Pf/ Soon. □

Def. (homotopic to a constant mapping)

Cor. $f : M \to N$ null homotopic $\Longrightarrow f^* : H^k N \to H^k M$ is the zero mapping for $k \geq 1$.

Pf/ Apply the homotopy invariance theorem, and see Prop. ☆, above. □

Def. ($id_M \sim$ constant mapping, i.e. id_M null homotopic)

Cor. M contractible $\Longrightarrow H^k M = 0$ for $k \geq 1$.

Pf/ $id_M^* : H^k M \to H^k M$ is the identity mapping. Apply the previous corollary. □

Cor. $H^k \mathbb{R}^n = 0$ for $k \geq 1$.

Pf/ $h : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$, $(t, x) \mapsto tx$ is a contraction, i.e., a homotopy between the identity mapping and a constant mapping. $\square$

Cor. $M \subseteq \mathbb{R}^3$ contractible, $F : M \to TM$ a vector field on M (a section of the tangent bundle).

(1) $\operatorname{curl} F = 0 \Rightarrow \exists$ potential $\phi : M \to \mathbb{R}$ s.t. $F = \operatorname{grad} \phi$.

(2) $\operatorname{div} F = 0 \Rightarrow \exists$ vector fld. G s.t. $F = \operatorname{curl} G$

(3) $g : M \to \mathbb{R} \Rightarrow \exists$ vector fld. G s.t. $\operatorname{div} G = g$.

Pf/ See the previous lecture. $\square$