Math 411

M oriented n-manifold with boundary

$\omega \in \Omega^{n-1} M$ with compact support

Def. $\operatorname{supp}(\omega) = \overline{\{ p \in M : \omega(p) \neq 0 \}}$

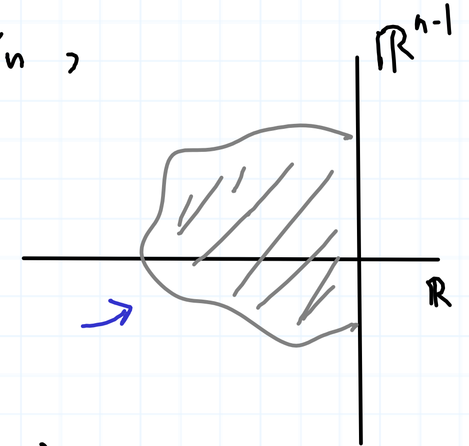
Thm. (Stokes' thm.)

really $\iota^* \omega$ where $\iota : \partial M \hookrightarrow M$

$$\int_M d\omega = \int_{\partial M} \omega .$$

omit

Pf/ Case 1 $\quad M = \mathbb{R}^n_-$, $\quad \omega = \sum_{i=1}^n a_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$,



supp(ω)

$$d\omega = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial a_i}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^n (-1)^{i+1} \frac{\partial a_i}{\partial x_i} \, dx_1 \wedge \cdots \wedge dx_n$$

Then $\int_M d\omega = \sum_{i=1}^n (-1)^{i+1} \int_{\mathbb{R}^n_-} \frac{\partial a_i}{\partial x_i}$. To compute $\int_{\mathbb{R}^n_-} \frac{\partial a_i}{\partial x_i}$,

use Fubini's thm. to integrate with respect to x_i first.

For $i \neq 1$, we have $\int_{x_i=-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} = \lim_{t \to \infty} \int_0^t \frac{\partial a_i}{\partial x_i} + \lim_{t \to -\infty} \int_t^0 \frac{\partial a_i}{\partial x_i}$

$$= \lim_{t\to\infty} \Big(a_i(x_1, \dots, \underset{\substack{\uparrow \\ i^{\text{th}} \text{ spot}}}{t}, \dots, x_n) - a_i(x_1, \dots, 0, \dots, x_n) \Big) + \lim_{t \to -\infty} \Big(a_i(x_1, \dots, 0, \dots, x_n) - a_i(x_1, \dots, 0, \dots, x_n) \Big)$$

$$= [\underset{*}{0} - a_i(x_1, \dots, 0, \dots, x_n)] + [a_i(x_1, \dots, 0, \dots, x_n) - \underset{*}{0}] = 0$$

$*$ since supp ω is compact

For $i = 1$, we have $\int_{x_1=-\infty}^{0} \frac{\partial a_1}{\partial x_1} = \lim_{t\to-\infty} \int_{-\infty}^{0} \frac{\partial a_1}{\partial x_1} = a_1(0, x_2, \dots, x_n)$.

Thus, $\int_M d\omega = \sum_{i=1}^n (-1)^{i+1} \int_{\mathbb{R}^n_-} \frac{\partial a_i}{\partial x_i} = \int_{\mathbb{R}^n_-} \frac{\partial a_1}{\partial x_1} = \int_{\mathbb{R}^{n-1}} \int_{x_1=-\infty}^{0} \frac{\partial a_1}{\partial x_1}$

$$= \int_{\mathbb{R}^{n-1}} a_1(0, x_2, \dots, x_n).$$

One the other hand, $\iota: \partial M \to M$

$(x_2, \dots, x_n) \mapsto (0, x_2, \dots, x_n)$

Note: $z_1 = 0$

$z_2 = x_2$

$\vdots$

$z_n = x_n$

$$\int_{\partial M} \omega = \int_{\partial M} \iota^* \omega = \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} \iota^* (a_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n)$$

$$= \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} a_i(0, x_2, \dots, x_n) \, d\iota_1 \wedge \cdots \wedge \widehat{d\iota_i} \wedge \cdots \wedge d\iota_n$$

$$= \int_{\mathbb{R}^{n-1}} a_1(0, x_2, \dots, x_n) \, dx_2 \wedge \cdots \wedge d_n = \int_{\mathbb{R}^{n-1}} a_1(0, x_2, \dots, x_n) .$$

Case 2 $\quad$ $\operatorname{supp} \omega \subset U$ for some orienting chart (U, h).

Use the chart to reduce to the case of $M = h(U) =$ open subset of $\mathbb{R}^n_-$.

But since ω is compact, we can extend ω to $\tilde{\omega}$ on all of $\mathbb{R}^n_-$ by letting $\tilde{\omega}|_U = \omega$ and $\tilde{\omega}|_{U^c} = 0$. Then $\int_{\mathbb{R}^n_-} \tilde{\omega} = \int_M \omega$.

This reduces the problem to case 1.

Case 3 $\quad$ General case $\quad$ For each $p \in M$, choose $\lambda_p : M \to [0,1]$ (C^∞), with $\operatorname{supp} \lambda_p$ compact and contained in U_p for some orienting chart (U_p, h_p) at p and such that $\lambda_p(p) > 0$. (See our text: existence of partitions of unity, or see the wiki page for bump functions.)

Then $\{\lambda_p^{-1}((0,1])\}_{p \in M}$ is an open cover of $\operatorname{supp}(\omega)$, which is compact. Thus, $\exists \ p_1, \dots, p_k \in M$ s.t. $\{\lambda_{p_i}^{-1}((0,1])\}_{i=1}^{k}$ covers $\operatorname{supp}(\omega)$.

Define $X := \bigcup_{i=1}^{k} \lambda_{p_i}^{-1}((0,1])$, an open subset of M containing $\operatorname{supp}(\omega)$.

Define

$$\tau_i : X \longrightarrow [0,1]$$

$$x \longmapsto \frac{\lambda_{p_i}(x)}{\sum_{j=1}^{k} p_j(x)}$$

"Partition of unity" since $\sum_{i=1}^{k} \tau_i(x) = 1 \quad \forall x \in X$.

for $i = 1, \dots, k$.

Define $\omega_i \in \Omega^{n-1} M$ by

$$\omega_{i_p} = \begin{cases} T_i(p)\, \omega_p & \text{for } p \in X \\ 0 & \text{otherwise.} \end{cases}$$

for $i = 1, \ldots, k$. Then each ω_i has compact support and is differentiable on M, and $\omega = \omega_1 + \cdots + \omega_k$.

We have $\operatorname{supp}(\omega_i) \subseteq U_{p_i}$, so by the previous case, $\int_{\partial M} \omega_i = \int_M d\omega_i$ for each i. We get $\int_{\partial M} \omega = \int_M d\omega$ by linearity. $\square$