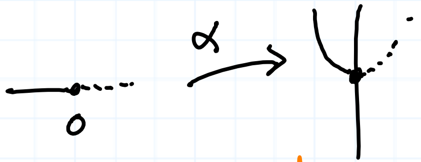
M an n-manifold w/ boundary.

Note: T_pM makes sense when defined as before in any of its guises: $T_p^{geom}M$, $T_p^{alg}M$, $T_p^{phys}M$, even if $p \in \partial M$:

geom: curves



α extends locally near $t=0$

alg: derivations $\xi_p \to \mathbb{R}$

A germ at p is required to extend locally to nbd. of p

phys: charts at $p \longrightarrow \mathbb{R}^n$

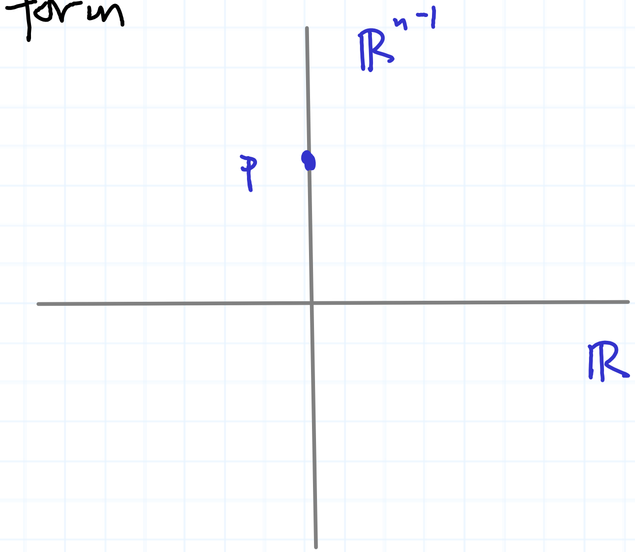
Define as before

$$v(U,h) \xrightarrow{(k\circ h^{-1})'(hp)} v(V,k)$$

Lemma. Suppose U, V are open subsets of $\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_1 \le 0\}$, and let $f: U \to V$ be a diffeomorphism. Let $p \in \partial U$. Then $df_p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ maps $\mathbb{R}^n_-$ to $\mathbb{R}^n_-$, maps $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$, and maps $\partial\mathbb{R}^n_-$ to $\partial\mathbb{R}^n_-$.

Pf/ The diffeo extends to a differentiable function in a neighborhood of p. So we may use f to denote this local extension, if necessary.

Consider the Jacobian of f at p in block form



$$Jf_p = \left[\begin{array}{c|ccc} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_1}{\partial x_2}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \hline * & & & \\ \vdots & & Jf|_{\partial \mathbb{R}^n_-}(p) & \\ * & & & \end{array}\right]$$

Recall: $\frac{\partial f_i}{\partial x_j} = \lim_{t \to 0} \frac{f_i(p+te_j) - f_i(p)}{t}$.

We have shown that $f(\partial U) = \partial V$. Hence, for $i=1$ and $j \geq 2$,

$$\frac{\partial f_1}{\partial x_j}(p) = \lim_{t \to 0} \frac{f_1(p+te_j) + f_1(p)}{t} = \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$

Further, for $t \leq 0$, $f_1(p+te_1) \leq 0$. Hence,

Since $f(U) \subseteq V \subseteq \mathbb{R}^n_-$

$$\frac{\partial f_1}{\partial x_1}(p) = \lim_{t \to 0^-} \frac{f_1(p+te_1) - f_1(p)}{t} = \lim_{t \to 0^-} \frac{\overbrace{f_1(p+te_1)}^{\leq 0} - 0}{\underbrace{t}_{\leq 0}} \geq 0$$

So $\frac{\partial f_1}{\partial x_1}(p) \geq 0$. But, in fact, $\frac{\partial f_1}{\partial x_1}(p) > 0$ since $\det Jf_p \neq 0$. The Jacobian thus has the form

$$Jf_p = \left[\begin{array}{c|ccc} + & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & Jf|_{\partial \mathbb{R}^n_-}(p) & \\ * & & & \end{array}\right],$$

from which the result follows. $\square$

For $p \in \partial M$, pick any chart (U,h) at p. We get $T_pM \xrightarrow{\cong} \mathbb{R}^n$ $(\star)$.
$v \mapsto v(U,h)$

Define $T_p^+M := \{v \in T_pM : v_1 \geq 0\}$ and $T_p^-M := \{v \in T_pM : v_1 \leq 0\}$. By the preceding lemma, these definitions don't depend on the choice of chart.

There is an inclusion of manifolds $\partial M \hookrightarrow M$ and $T(\partial M) = T_p^+ M \cap T_p^- M$.

Define $T_p^+ M \setminus T_p(\partial M) :=$ outward-pointing tangent vectors

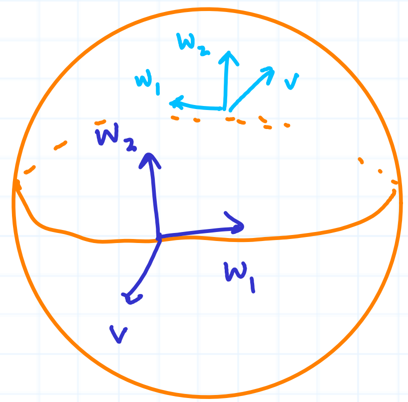
$T_p^- M \setminus T_p(\partial M) :=$ inward-pointing tangent vectors.

Interesting that this can be done given that M is not embedded in space.

If M is oriented, there is a natural orientation for ∂M: an ordered basis $w_1, \dots, w_{n-1}$ for $T_p \partial M$ is positive if $v, w_1, \dots, w_{n-1}$ is a positive ordered basis for $T_p M$ for any outward-pointing tangent vector v.

Example B_3 = solid unit ball in $\mathbb{R}^3$ with orientation induced by $\mathbb{R}^3$.

$\partial B_3 = S^2$



$\mathrm{Alt}^l V$

Recall $\mathrm{Alt}^l V$ = alternating l-linear maps $V \times \cdots \times V \to k$

Universal property:

$$\begin{tikzcd} & \Lambda^l V \arrow[d, dashed, "\exists!\, \tilde{\omega} \in (\Lambda^l V)^*"] \\ V \times \cdots \times V \arrow[ur] \arrow[r, "\omega"] & k \end{tikzcd}$$

$$(\Lambda^l V)^* \xleftrightarrow{\ \cong\ } \Lambda^l V^*$$

$$\eta \longleftarrow\!\shortmid\ \varphi_1 \wedge \cdots \wedge \varphi_l$$

$$\eta(v_1, \ldots, v_l) = \det\left(\varphi_i(v_j)\right)$$

So we identify $\mathrm{Alt}^l V$, $(\Lambda^l V)^*$, and $\Lambda^l V^*$.

Def. $\mathrm{Alt}\, V := \bigoplus_{l \geq 0} \mathrm{Alt}^l V$, $\quad \Lambda V^* := \bigoplus_{l \geq 0} \Lambda^l V^*$

k-algebra structure

$$\Lambda^r V^* \times \Lambda^s V^* \longrightarrow \Lambda^{r+s} V^*$$
$$(\omega, \eta) \longmapsto \omega \wedge \eta$$

Prop. $(\omega \wedge \eta)(v_1, \ldots, v_{r+s}) = \frac{1}{r!\,s!} \sum_{\tau \in S_{r+s}} \mathrm{sgn}(\tau)\, \omega(v_{\tau(1)}, \ldots, v_{\tau(r)})\, \eta(v_{\tau(r+1)}, \ldots, v_{\tau(r+s)})$.

$\uparrow$ symmetric group

Pf/ Exercise.

Characterization of $\wedge$

$$\Lambda V^* \times \Lambda V^* \to \Lambda V^* \qquad (\text{or} \quad \mathrm{Alt}\, V \times \mathrm{Alt}\, V \to \mathrm{Alt}\, V)$$

is the only operation that is

0. graded: $\quad \Lambda^r V^* \times \Lambda^s V^* \to \Lambda^{r+s} V^*$

1. (i) bilinear

   (ii) associative

   (iii) anti-commutative: for $\omega \in \Lambda^r V^*$, $\eta \in \Lambda^s V^*$, we have $\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega$

   (iv) $1 \in \Lambda^0 V^* \cong k$ acts as $1 \wedge \omega = \omega \quad \forall \omega \in \Lambda V^*$.

2. natural: For $f : V \to W$, $\omega, \eta \in \Lambda W^*$, we have

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta).$$

3. $(e_1^* \wedge \cdots \wedge e_n^*)(e_1 \wedge \cdots \wedge e_n) = 1$ for any basis $e_1, \ldots, e_n$ for V.

Summary $\Lambda : V \longmapsto \Lambda V^*$ is a contravariant functor

from the category of vector spaces/k to the category of graded anti-commutative algebras/k.

$$
\begin{tikzcd}
U \arrow[r] & V \arrow[dl] \\
W \arrow[u] &
\end{tikzcd}
\quad \overset{\Lambda}{\longleftrightarrow} \quad
\begin{tikzcd}
\Lambda U^* \arrow[d] & \Lambda V^* \arrow[l] \\
\Lambda W^* \arrow[ur] &
\end{tikzcd}
$$