

Integration

M an oriented manifold, $\omega \in \Omega^n M$

$X \subseteq M$ is measurable iff
 $h(X \cap U) \subseteq \mathbb{R}^n$ is measurable for
 all charts (U, h)

To define the integral, cover M by disjoint measurable sets $\{A_i\}_{i=1}^{\infty}$
 and orientation preserving charts $\{(U_i, h_i)\}$ with $A_i \subseteq U_i$.

Locally, with respect to (U_i, h_i) ,

$$\omega(p) = \tilde{a}_i(p) dx_{1,p} \wedge \dots \wedge dx_{n,p}$$

where $\tilde{a}_i : U_i \rightarrow \mathbb{R}$. Define $a_i : h_i(U_i) \rightarrow \mathbb{R}$ by $a_i = \tilde{a}_i \circ h_i^{-1}$.

Then ω is **integrable** if each a_i is Lebesgue integrable on $h(A_i)$
 and $\sum_{i=1}^{\infty} \int_{h(A_i)} |a_i| < \infty$. In this case

$$\int_M \omega := \sum_{i=1}^{\infty} \int_{h(A_i)} a_i.$$

Claims

(2)

- ① The collections $\{A_i\}$, $\{(U_i, A_i)\}$ with the claimed properties (A_i measurable, $A_i \subseteq U_i$, h_i orientation preserving) exists.
- ② The integral is independent of the choices made.

- ① M oriented $\Rightarrow \exists$ orienting atlas $\{(V_\alpha, k_\alpha)\}$.
 M second-countable $\Rightarrow \exists$ countable basis \mathcal{B} for the topology.
For each $p \in M$, take a chart (V_α, k_α) at p . Find $U \in \mathcal{B}$ with $U \subseteq V_\alpha$. Save the chart (U, h_U) where $h_U = k_\alpha|_U$.
In the end, we have a countable orientation preserving atlas $\{(U_i, h_i)\}$.

Now let $A_1 = U_1$ and $A_{i+1} = U_i \setminus \bigcup_{k=1}^i A_k$ for $i \geq 1$.

② \int is independent of choices

③

Let $\{A_i\}$, $\{(U_i, h_i)\}$, α_i 's be as above, with α_i 's integrable and $\sum_{i=1}^{\infty} \int_{h_i(A_i)} |\alpha_i| < \infty$. Suppose $\{B_i\}_{i=1}^{\infty}$ disjoint, measurable,

$\{(V_i, k_i)\}_{i=1}^{\infty}$ an orientation preserving atlas, $B_i \subseteq V_i$,

w/ $V_i = b_i dy_1 \wedge \dots \wedge dy_n$ w.r.t. (V_i, k_i) .

To show: (i) b_i integrable on $k_i(B_i) \forall i$.

(ii) $\sum_{i=1}^{\infty} \int_{k_i(B_i)} |b_i| < \infty$.

(iii) $\sum_i \int_{h_i(A_i)} \alpha_i = \sum_i \int_{k_i(B_i)} b_i$

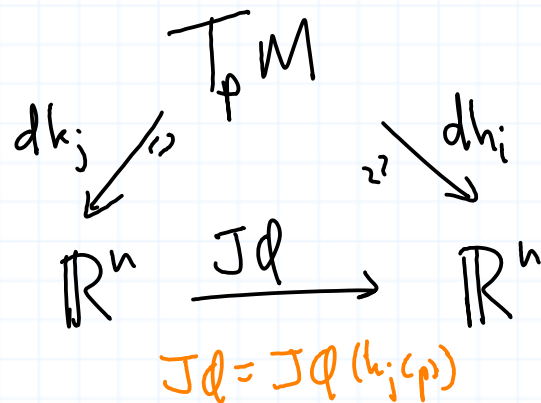
Idea

$$\begin{aligned}
 \int_M \omega &= \sum_{i=1}^{\infty} \int_{h_i(A_i)} a_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{h_i(A_i \cap B_j)} a_i \\
 &\stackrel{\text{change of variables}}{=} \sum_i \sum_j \int_{k_j(A_i \cap B_j)} b_j \\
 &= \sum_j \sum_i \int_{k_j(A_i \cap B_j)} b_j \\
 &= \sum_j \int_{k_j(B_j)} b_j.
 \end{aligned}$$

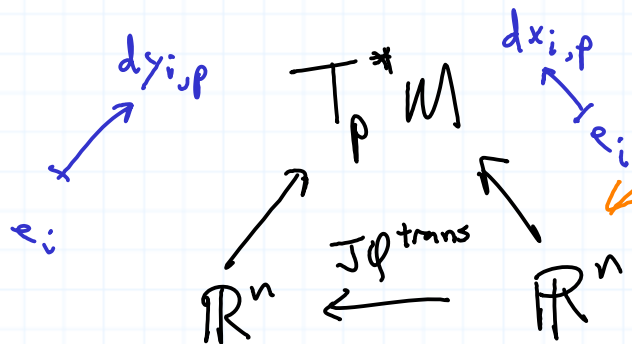
We will look at the change of variables step.

$$\int_{h_i(A_i \cap B_j)} a_i = \int_{k_j(A_i \cap B_j)} b_j$$

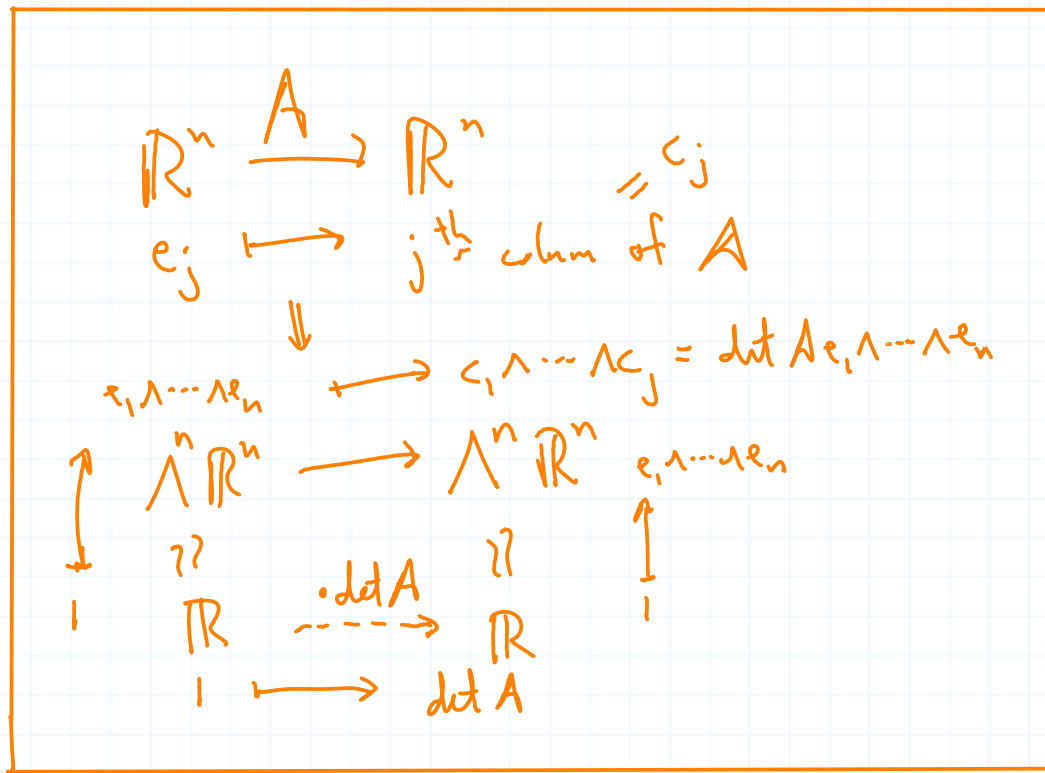
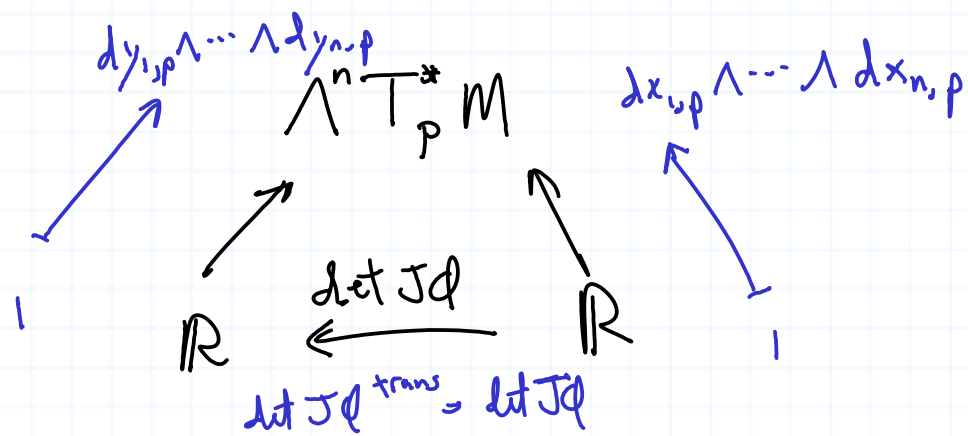
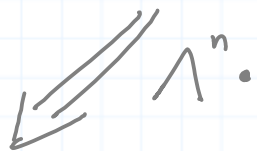
For $p \in A_i \cap B_j \subseteq U_i \cap V_j$, we have



Dualizing:



(here $\mathbb{R}^n \cong (\mathbb{R}^n)^*$
 $e_i \leftrightarrow e_i^*$)



Thus, by commutativity,

$$(\star) \quad dx_{1,p} \wedge \dots \wedge dx_{n,p} = \det J\phi \, dy_{1,p} \wedge \dots \wedge dy_{n,p} \quad (\star)$$

In local coordinates, $\omega_p = \tilde{a}_i(p) dx_{1,p} \wedge \dots \wedge dx_{n,p} = \tilde{b}_j(p) dy_{1,p} \wedge \dots \wedge dy_{n,p}$. (6)

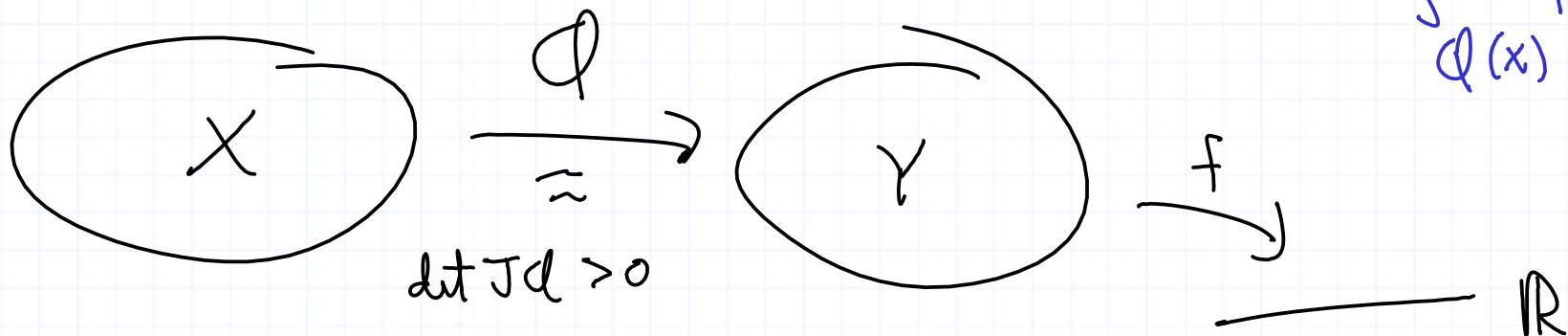
By $(*)$, we get $\tilde{a}_i(p) dx_{1,p} \wedge \dots \wedge dx_{n,p} = \tilde{a}_i(p) \det JQ dy_{1,p} \wedge \dots \wedge dy_{n,p}$.

Hence, $\tilde{a}_i(p) \det JQ(k_j(p)) = \tilde{b}_j(p)$. Therefore,

$$\begin{aligned} b_j(k_j(p)) &= \tilde{b}_j(p) = \tilde{a}_i(p) \det JQ(k_j(p)) = a_i(h(p)) \det JQ(k_j(p)) \\ &= (a_i \circ Q)(k_j(p)) \det JQ(k_j(p)), \end{aligned}$$

i.e. $b_j = (a_i \circ Q) \det JQ$.

Chain rule (picture version)



$$\int_{Q(x)} f = \int_x f \circ Q \det JQ$$

Putting everything together:

$$\int_{h_i(A_i \cap B_j)} a_i = \int_{\mathcal{Q}} \overset{C \text{ of } V}{k_j(A_i \cap B_j)} a_i = \int_{k_j(A_i \cap B_j)} a_i \circ \mathcal{Q} \text{ det } J \mathcal{Q}$$

$$= \int_{k_j(A_i \cap B_j)} b_j.$$

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