# Math 411

## Integration

$M$ an oriented manifold, $\omega \in \Omega^n M$

$X \subseteq M$ is measurable iff $h(X \cap U) \subseteq \mathbb{R}^n$ is measurable for all charts $(U, h)$

To define the integral, cover $M$ by disjoint measurable sets $\{A_i\}_{i=1}^{\infty}$ and orientation preserving charts $\{(U_i, h_i)\}$ with $A_i \subseteq U_i$.

Locally, with respect to $(U_i, h_i)$,

$$\omega(p) = \tilde{a}_i(p)\, dx_{1,p} \wedge \cdots \wedge dx_{n,p}$$

where $\tilde{a}_i : U_i \to \mathbb{R}$. Define $a_i : h_i(U_i) \to \mathbb{R}$ by $a_i = \hat{a}_i \circ h_i^{-1}$.

Then $\omega$ is integrable if each $a_i$ is Lebesgue integrable on $h(A_i)$ and $\sum_{i=1}^{\infty} \int_{h(A_i)} |a_i| < \infty$. In this case

$$\int_M \omega := \sum_{i=1}^{\infty} \int_{h(A_i)} a_i .$$

## Claims

1. The collections $\{A_i\}$, $\{(U_i, A_i)\}$ with the claimed properties ($A_i$ measurable, $A_i \subseteq U_i$, $h_i$ orientation preserving) exists.

2. The integral is independent of the choices made.

1. $M$ oriented $\Rightarrow \exists$ orienting atlas $\{(V_\alpha, k_\alpha)\}$.
   $M$ second-countable $\Rightarrow \exists$ countable basis $\mathcal{B}$ for the topology.
   For each $p \in M$, take a chart $(V_\alpha, k_\alpha)$ at $p$. Find $U \in \mathcal{B}$ with $U \subseteq V_\alpha$. Save the chart $(U, h_V)$ where $h_V = k_\alpha | V$.
   In the end, we have a countable orientation preserving atlas $\{(U_i, h_i)\}$.

Now let $A_1 = U_1$ and $A_{i+1} = U_i \setminus \bigcup_{k=1}^{i} A_i$ for $i \geq 1$.

## 2 $\int$ is independent of choices

Let $\{A_i\}$, $\{(U_i, h_i)\}$, $a_i$'s be as above, with $a_i$'s integrable and $\sum_{i=1}^{\infty} \int_{h(A_i)} |a_i| < \infty$. Suppose $\{B_i\}_{i=1}^{\infty}$ disjoint, measurable, $\{(V_i, k_i)\}_{i=1}^{\infty}$ an orientation preserving atlas, $B_i \subseteq V_i$,

$w|_{V_i} = b_i \, dy_1 \wedge \cdots \wedge dy_n$ w.r.t. $(V_i, k_i)$.

To show:

(i) $b_i$ integrable on $k_i(B_i)$ $\forall i$.

(ii) $\sum_{i=1}^{\infty} \int_{k_i(B_i)} |b_i| < \infty$.

(iii) $\sum_i \int_{h_i(A_i)} a_i = \sum_i \int_{k_i(B_i)} b_i$

## Idea

$$\int_M \omega = \sum_{i=1}^{\infty} \int_{h_i(A_i)} a_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{h_i(A_i \cap B_j)} a_i$$

$$\overset{\text{change of variables}}{=} \sum_i \sum_j \int_{k_j(A_i \cap B_j)} b_j$$

$$= \sum_j \sum_i \int_{k_j(A_i \cap B_j)} b_j$$

$$= \sum_j \int_{k_j(B_j)} b_j .$$

We will look at the change of variables step.

$$\int_{h_i(A_i \cap B_j)} a_i = \int_{k_j(A_i \cap B_j)} b_j$$

For $p \in A_i \cap B_j \subseteq U_i \cap V_j$, we have

$$\begin{tikzcd}
& T_pM \arrow[dl, "dk_j"', "\cong"] \arrow[dr, "dh_i", "\cong"'] & \\
\mathbb{R}^n \arrow[rr, "J\varphi"] & & \mathbb{R}^n
\end{tikzcd}$$

$\varphi := h_i \circ k_j^{-1}$

$J\varphi = J\varphi(k_j(p))$

Dualizing:

$$\begin{tikzcd}[column sep=small]
 & T_p^* M & \\
\mathbb{R}^n \arrow[ur] & & \mathbb{R}^n \arrow[ul] \arrow[ll, "J\varphi^{trans}"']
\end{tikzcd}$$

$e_i \mapsto dy_{i,p}$ $\qquad$ $e_i \mapsto dx_{i,p}$

(Here $\mathbb{R}^n \simeq (\mathbb{R}^n)^*$
$e_i \leftrightarrow e_i^*$

$$\Big\Downarrow \Lambda^n .$$

$$\begin{tikzcd}[column sep=small]
 & \Lambda^n T_p^* M & \\
\mathbb{R} \arrow[ur] & & \mathbb{R} \arrow[ul] \arrow[ll, "\det J\varphi"']
\end{tikzcd}$$

$1 \mapsto dy_{1,p} \wedge \cdots \wedge dy_{n,p}$ $\qquad$ $1 \mapsto dx_{1,p} \wedge \cdots \wedge dx_{n,p}$

$\det J\varphi^{trans} = \det J\varphi$

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$$

$e_j \longmapsto j^{th}$ column of $A$ $= c_j$

$$\Downarrow$$

$e_1 \wedge \cdots \wedge e_n \longmapsto c_1 \wedge \cdots \wedge c_j = \det A \, e_1 \wedge \cdots \wedge e_n$

$$\begin{tikzcd}
\Lambda^n \mathbb{R}^n \arrow[r] & \Lambda^n \mathbb{R}^n \\
\mathbb{R} \arrow[u, "\wr"] \arrow[r, dashed, "\cdot \det A"] & \mathbb{R} \arrow[u, "\wr"']
\end{tikzcd}$$

$1 \mapsto e_1 \wedge \cdots \wedge e_n$ $\qquad$ $1 \longmapsto \det A$ $\qquad$ $1 \mapsto e_1 \wedge \cdots \wedge e_n$

Thus, by commutativity,

$$(\circledast) \qquad dx_{1,p} \wedge \cdots \wedge dx_{n,p} = \det J\varphi \; dy_{1,p} \wedge \cdots \wedge dy_{n,p} \qquad (\circledast)$$

In local coordinates, $\omega_p = \tilde{a}_i(p)\, dx_{1,p} \wedge \cdots \wedge dx_{n,p} = \hat{b}_j(p)\, dy_{1,p} \wedge \cdots \wedge dy_{n,p}$.

By (*), we get $\tilde{a}_i(p)\, dx_{1,p} \wedge \cdots \wedge dx_{n,p} = \tilde{a}_i(p) \det J\varphi \; dy_{1,p} \wedge \cdots \wedge dy_{n,p}$.

Hence, $\tilde{a}_i(p) \det J\varphi\,(k_j(p)) = \hat{b}_j(p)$. Therefore,

$$b_j(k_j(p)) = \hat{b}_j(p) = \tilde{a}_i(p) \det J\varphi(k_j(p)) = a_i(h(p)) \det J\varphi\,(k_j(p))$$

$$= (a_i \circ \varphi)(k_j(p)) \det J\varphi\,(k_j(p)),$$

i.e. $b_j = (a_i \circ \varphi) \det J\varphi$.

Chain rule (picture version)

$$\int_{\varphi(X)} f = \int_X f \circ \varphi \det J\varphi$$

$$X \xrightarrow[\cong,\ \det J\varphi > 0]{\varphi} Y \xrightarrow{f} \mathbb{R}$$

Putting everything together:

$$\int_{h_i(A_i\cap B_j)} a_i = \int_{\varphi k_j(A_i\cap B_j)} a_i \overset{\text{C of V}}{=} \int_{k_j(A_i\cap B_j)} a_i\circ\varphi \det J\varphi$$

$$= \int_{k_j(A_i\cap B_j)} b_j .$$