Reading / Return HW

Oriented Manifolds

An oriented manifold is a manifold M with a collection of orientations $\mathcal{O} = \{\mathcal{O}_p\}_{p\in M}$ with $\mathcal{O}_p$ an orientation on T_pM such that $\mathcal{O}$ is locally coherent, meaning that for each point of M, there is a chart (U,h) at that point such that $\forall p \in U$, the isomorphism induced by the chart:

$$\begin{aligned} T_pM &\to \mathbb{R}^n \\ v &\mapsto v(U,h) \end{aligned}$$

takes $\mathcal{O}_p$ to the usual positive orientation of $\mathbb{R}^n$.

A diffeomorphism $f: M \to N$ is orientation preserving if $df_p: T_pM \to T_pN$ is orientation preserving for all $p\in M$.

A chart (U, h) is orientation preserving if $\forall p \in U$, the isomorphism

$$T_pM \longrightarrow \mathbb{R}^n$$

takes $\mathcal{O}_p$ to the positive orientation of $\mathbb{R}^n$.

An atlas $\mathcal{U} = \{(U, h)\}$ is an orienting atlas if all the transition functions are orientation preserving, i.e., their derivatives have positive determinant. In this case, $\exists!$ orientation of M such that $\mathcal{U}$ consists of orientation preserving charts.

Example

* S^n is orientable
* a Möbius strip is not orientable
* $\mathbb{P}^n$ is orientable iff n is odd (See HW.)

Integration on Manifolds

Basic idea Let M be an n-manifold, $\omega \in \Omega^n M$ (a section of $\Lambda^n T^*M$).

Choosing coordinates (U, h), we get a local expression for ω:

$$\omega(p) = \sum \tilde{a}(p)\, dx_{1,p} \wedge \cdots \wedge dx_{n,p}.$$

① If $A \subseteq U$, we let $\int_A \omega = \int_{h(A)} a$ where $a = \tilde{a} \circ h^{-1}$.

② If $A \subseteq V$ for some other chart (V, k), it turns out that we'll get the same value for the integral!

③ To define $\int_M \omega$, divide M up into nice disjoint pieces like A, above, which fit inside charts.

Note: The existence of an orientation must be relevant.