Math 411

Last time $f: M \to \mathbb{R} \implies df_p : T_pM \to \mathbb{R}$, i.e. $df \in T_p^*M$.

In local coords.,

really $\frac{\partial(f\circ h^{-1})}{\partial x_i}(p)$

$$(*) \qquad df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\, dx_{i,p}\,.$$

For general $f: M \to N$ we get $f_{*,p} := df_p : T_pM \to T_{f(p)}N$ and, thus, $f_p^* := df_p^* : T_{f(p)}^*N \to T_p^*M$ and $f_p^* : \Lambda^k T_{f(p)}^*N \to \Lambda^k T_p^*M$. In coords. (U,h) at p, (V,k) at $f(p)$ with $U \subseteq f^{-1}(V)$:

$$
\begin{tikzcd}
\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \ @\ p \arrow[d, mapsto] & T_pM \arrow[r, "df_p"] \arrow[d] & T_{f(p)}N \arrow[d] & \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \ @\ f(p) \\
e_1, \dots, e_m & \mathbb{R}^m \arrow[r, "``\left(\frac{\partial f_i}{\partial x_j}(p)\right)\text{''}", "(k\circ f\circ h^{-1})'(h(p))"'] & \mathbb{R}^n & \tilde{e}_1, \dots, \tilde{e}_n
\end{tikzcd}
$$

$$
\begin{tikzcd}
dy_1, \dots, dy_n \ @\ p \arrow[d, mapsto] & T^*_{f(p)}N \arrow[r, "f^*_p"] \arrow[d] & T^*_pM \arrow[d] & dx_1, \dots, dx_m \ @\ p \arrow[d, mapsto] \\
\tilde{e}_1^*, \dots, \tilde{e}_n^* \arrow[d, mapsto] & (\mathbb{R}^n)^* \arrow[r] \arrow[d, phantom, "\wr"] & (\mathbb{R}^m)^* \arrow[d, phantom, "\wr"] & \tilde{e}_1^*, \dots, \tilde{e}_m^* \arrow[d, mapsto] \\
\tilde{e}_1, \dots, \tilde{e}_n & \mathbb{R}^n \arrow[r, "\left(\frac{\partial f_i}{\partial x_j}(p)\right)^{\text{trans}}"] & \mathbb{R}^m & e_1, \dots, e_m
\end{tikzcd}
$$

Thus, $df_p^*(dy_i)$ "=" i^{th} column of $\left(\frac{\partial f_i}{\partial x_j}(p)\right)^{\text{trans}}$

$= i^{\text{th}}$ row of $\left(\frac{\partial f_i}{\partial x_j}(p)\right) = \nabla f_i\,(p)$

"=" $\sum_j \frac{\partial f_i}{\partial x_j}(p)\, dx_j = df_{i,p}$

$$df_p^*(dy_i) = df_{i,p}$$

Thus, $f_p^* : \Lambda^k_{f(p)} T^* N \longrightarrow \Lambda^k T_p^* M$ is given locally by

$$f_p^*\Big(\sum_\mu w_\mu(f(p))\, dy_{\mu_1} \wedge \cdots \wedge dy_{\mu_k}\Big) = \sum_\mu w_\mu(f(p))\, df_{\mu_1} \wedge \cdots \wedge df_{\mu_k}.$$

Bundles

$f: M \to N$ induces mappings of bundles

$$f_*: TM \to TN \qquad\qquad f^*: \Omega^k N \to \Omega^k M.$$

For instance, in local coords (U,h), (V,k), as usual,

$$\begin{tikzcd}
\pi_M^{-1}(U) \arrow[r, "f_*"] \arrow[d, phantom, "\wr"] & \pi_N^{-1}(V) \arrow[d, phantom, "\wr"] \\
h(U) \times \mathbb{R}^m \arrow[r] & k(V) \times \mathbb{R}^n
\end{tikzcd}$$

$$(h(p), v) \longmapsto (k(f(p)), f_{*,p}(v))$$

Point: The mappings we defined locally (at each point p, above) glue together to give (smooth) mappings of bundles.

Example $\quad M = \mathbb{R}^2, \quad N = \mathbb{R}^3$

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (u,v) &\longmapsto (u^2 - v,\ u + 2v,\ v^2) \end{aligned}$$

$$\omega = \overbrace{x^2}^{\omega_{xy}(x,y,z)} dx \wedge dy + \overbrace{(x+z)}^{\omega_{yz}(x,y,z)} dy \wedge dz$$

$\in \Omega^2 \mathbb{R}^3$ = sections of $\Lambda^2 T^* \mathbb{R}^3$

$$f^*(\omega) = \overbrace{u^2 - v}^{\omega_{xy}(f(u,v))} \, d(u^2 - v) \wedge d(u + 2v) + \overbrace{(u^2 - v + v^2)}^{\omega_{yz}(f(u,v))} \, d(u + 2v) \wedge d v^2$$

$$= (u^2 - v)(2u\,du - dv) \wedge (du + 2dv) + (u^2 - v + v^2)(du + 2dv) \wedge (2v\,dv)$$

↑ by (*), page 1

$$= \left[(u^2 - v)(4u + 1) + 2(u^2 - v + v^2) v \right] du \wedge dv = (\text{etc})\, du \wedge dv.$$

Orientation

Def. Let V be a real vector space. Two ordered bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ have the same orientation if the mapping

$$\begin{aligned} V &\longrightarrow V \\ v_i &\longmapsto w_i \end{aligned}$$

has positive determinant.

The property of having the same orientation defines an equivalence relation on the set of ordered bases for V with two equivalence classes. Each equivalence class is called an orientation on V. Having chosen an orientation $\mathcal{O}$, we get an oriented vector space, $(V, \mathcal{O})$.

If an ordered basis $(w_1, \ldots, w_n)$ is in $\mathcal{O}$, we say $(w_1, \ldots, w_n)$ is **positively oriented**; otherwise it's **negatively oriented**.

Example. $V = \mathbb{R}^3$

$$(e_1, e_2, e_3) \sim (e_2, e_3, e_1) \sim (e_3, e_1, e_2)$$

$$(e_2, e_1, e_3) \sim (e_1, e_3, e_2) \sim (e_3, e_2, e_1)$$

Also, for example, $(e_1 + e_2, e_2, e_3) \sim (e_1, e_2, e_3)$ since

$$\mathbb{R}^3 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \mathbb{R}^3$$

$$\begin{aligned} e_1 &\longmapsto e_1 + e_2 \\ e_2 &\longmapsto e_2 \\ e_3 &\longmapsto e_3 \end{aligned}$$

has positive determinant.