Math 411

* Return HW and discuss

Prop. (1) $(\Lambda^\ell V)^* \cong \Lambda^\ell (V^*)$

(2) $(\mathrm{Sym}^\ell V)^* \cong \mathrm{Sym}^\ell (V^*)$.

Sketch for part 1 /

The isomorphism:

$$\Lambda^\ell (V^*) \longrightarrow (\Lambda^\ell V)^*$$

$$\varphi_1 \wedge \cdots \wedge \varphi_\ell \longmapsto [\, v_1 \wedge \cdots \wedge v_\ell \longmapsto \det(\varphi_i(v_j)) \,]$$

$$\begin{bmatrix} \varphi_1(v_1) & \varphi_1(v_2) & \cdots & \varphi_1(v_\ell) \\ \varphi_2(v_1) & \varphi_2(v_2) & \cdots & \varphi_2(v_\ell) \\ \vdots & \vdots & & \vdots \\ \varphi_\ell(v_1) & \varphi_\ell(v_2) & \cdots & \varphi_\ell(v_\ell) \end{bmatrix}$$

The inverse mapping in coordinates:

If $e_1, \dots, e_n$ is a basis for V:

$$\begin{tikzcd} & \Lambda^\ell V \arrow[d, "\omega"] \\ V \times \cdots \times V \arrow[ur] \arrow[r] & \mathbb{R} \end{tikzcd}$$

$$e_{\mu_1} \wedge \cdots \wedge e_{\mu_\ell} \longmapsto \omega(e_{\mu_1}, \dots, e_{\mu_\ell})$$

$$(e_{\mu_1}, \dots, e_{\mu_\ell}) \longmapsto \omega(e_{\mu_1}, \dots, e_{\mu_\ell}) =: \omega_\mu$$

$\mu = (\mu_1, \dots, \mu_n)$

inverse:

$$\omega \in (\Lambda^\ell V)^*$$
$$\big\downarrow$$
$$\textstyle\sum_\mu \omega_\mu e_\mu^* \in \Lambda^\ell (V^*)$$

$e_\mu^* := e_{\mu_1}^* \wedge \cdots \wedge e_{\mu_\ell}^*$

$$\Lambda^\ell V^* \longrightarrow (\Lambda^\ell V)^*$$
$$\varphi_1 \wedge \cdots \wedge \varphi_\ell \longmapsto \left[v_1 \wedge \cdots \wedge v_\ell \longmapsto \det(\varphi_i(v_j)) \right]$$

For inverse, choose a basis $e_1, \ldots, e_n$ for V. We get the dual basis $e_1^*, \ldots, e_n^*$ for V^*. Then $\{ e_\mu := e_{\mu_1} \wedge \cdots \wedge e_{\mu_\ell} : \mu_1 < \cdots < \mu_\ell \}$ is a basis for $\Lambda^\ell V$. The inverse of the above mapping is

$$(\Lambda^\ell V)^* \longrightarrow \Lambda^\ell V^*$$
$$\omega : \Lambda^\ell V \to k \longmapsto \sum_{\substack{\mu: \\ 1 \le \mu_1 < \cdots < \mu_\ell \le n}} \omega(e_\mu)\, e_\mu^*$$

$$\left(e_\mu^* = e_{\mu_1}^* \wedge \cdots \wedge e_{\mu_\ell}^* \right)$$

Pullbacks

$L : V \to W$ induces, for each $l \geq 0$,

A $\quad L^* : \Lambda^l W^* \longrightarrow \Lambda^l V^* \;\cong\; (\Lambda^l V)^*$

$$\varphi_1 \wedge \cdots \wedge \varphi_l \mapsto L^*\varphi_1 \wedge \cdots \wedge L^*\varphi_l \longmapsto \left[v_1 \wedge \cdots \wedge v_l \mapsto \det(\varphi_i \circ L(v_j)) \right]$$

B $\quad L^* : (\Lambda^l W)^* \longrightarrow (\Lambda^l V)^*$

$$[w : \Lambda^l W \to k] \longmapsto \left[\begin{array}{l} L^* w : \Lambda^l V \to k \\ \quad v_1 \wedge \cdots \wedge v_l \mapsto w(Lv_1 \wedge \cdots \wedge Lv_l) \end{array} \right]$$

Tangent and Cotangent Bundles

Vector spaces of most interest on a manifold:

tangent space: T_pM cotangent space: T_p^*M

and related vector spaces: $\Lambda^k T_pM$, $\Lambda^k T_p^*M$, $\mathrm{Sym}^k TM$, $T_p^*M^{\otimes k}$, etc.

Picking a chart (U,h) gives an isomorphism (of vector spaces)

$$\tau : T_pM \to \mathbb{R}^n$$

and we define $\left(\frac{\partial}{\partial x_i}\right)_p$ by $\tau\left(\frac{\partial}{\partial x_i}\right)_p = e_i$. The dual basis for T_p^*M is $dx_{1,p}, \dots dx_{n,p}$. So, dropping the p for typesetting purposes,

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

Def. The **tangent bundle** is the disjoint union $TM := \bigsqcup_{p\in M} T_pM$.

It comes with the projection:

$$\begin{tikzcd} TM \arrow[d, "\pi"'] \\ M \end{tikzcd}$$

defined by $\pi(v) = p$ if $p \in T_pM$.

For each chart (U,h) on M, we get a chart $(\pi^{-1}(U), \tilde{h})$:

$$\tilde{h} : \pi^{-1}(U) \longrightarrow h(U) \times \mathbb{R}^n \quad \subseteq \mathbb{R}^n \times \mathbb{R}$$
$$v \in T_pM \longmapsto (h(p), v(U,h))$$

Define $A \subseteq TM$ to be open if $\tilde{h}(\pi^{-1}(U) \cap A)$ is open for each chart. These charts then define the manifold structure on TM.

Transition functions for TM

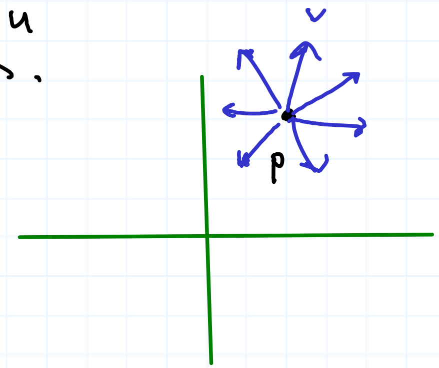
Let (U,h) and (V,k) be charts at $p \in M$. We have corresponding charts for TM:

$$\begin{tikzcd}[column sep=small]
& p \in T_pM \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& \pi^{-1}(U \cap V) \arrow[dl, "\tilde{h}"'] \arrow[dr, "\tilde{k}"] & \\
h(U\cap V) \times \mathbb{R}^n \arrow[rr, "{(k\circ h^{-1},\ (k\circ h^{-1})'(h(p)))}"] & & k(U \cap V) \times \mathbb{R}^n \\
(h(p), v(U,h)) \arrow[rr, mapsto] & & (k(p), v(V,k))
\end{tikzcd}$$

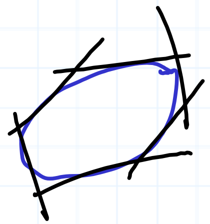
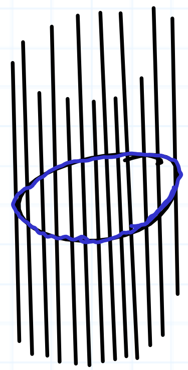
☆ **Notation:** One often denotes an element of TM as (p,v), meaning $v \in T_pM$.

Examples

* $M = \mathbb{R}^n$ Fix chart $(\mathbb{R}^n, id)$ for M, inducing
$$T_pM \xrightarrow{\cong} \mathbb{R}^n. \quad \text{Then } TM = \pi^{-1}(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n.$$



* $M = S^1$

 $\rightsquigarrow$  $\cong S^1 \times \mathbb{R}$

Note: Given M, N manifolds, there is a natural manifold structure on $M \times N$ (see our text). In general, $TM \not\cong M \times \mathbb{R}^n$. (Can you think of an example?)

However, given a chart (U, h) on M, we do have a diffeomorphism
$$\pi^{-1}(U) \cong U \times \mathbb{R}^n.$$

Similarly, we can define the cotangent bundle $T^*M := \dot{\bigcup}_p T_p^* M$ with its projection $\pi: T^*M \to M$ with its differential structure s.t. if (U, h) is a chart on M, then $\pi^{-1}(U) \cong U \times \mathbb{R}^n$, again.

In the same way, there are bundles $\Lambda^k TM$, $\Lambda^k T^*M$, $\mathrm{Sym}^k M$, $TM \otimes T^*M$, etc.