Math 411 (continued from last time)

$$T_p^{alg}(M) \longrightarrow T_p^{phys}(M)$$

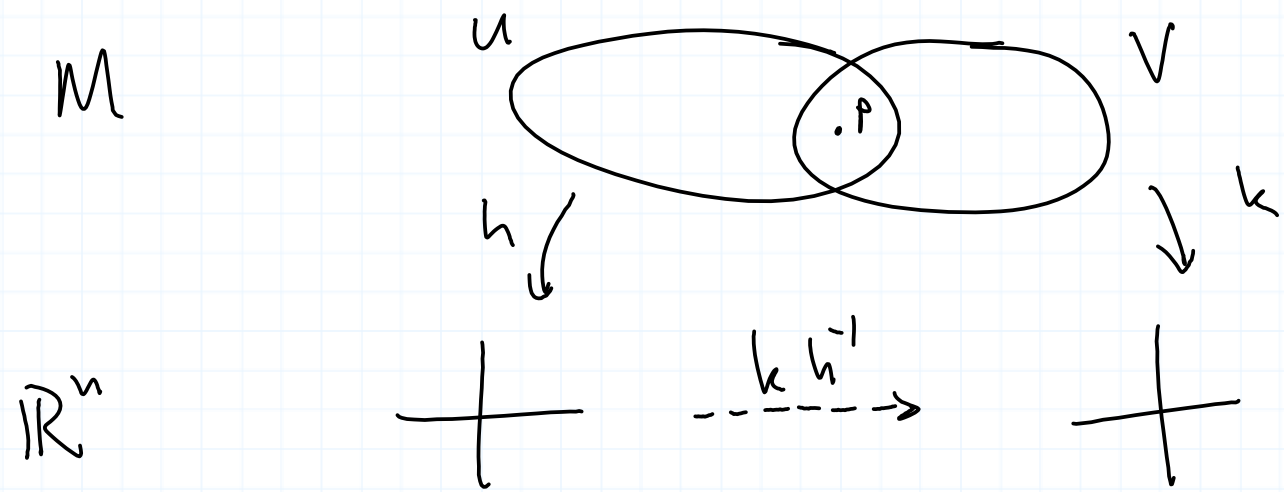
$v : \xi_p \to \mathbb{R}$ linear derivation $\qquad \bar{v} : D_p(M) \longrightarrow \mathbb{R}^n$

$$(U, h) \longmapsto (v(h_1), \dots, v(h_n))$$

where $h_i : U \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}$

$$x \longmapsto (h_1(x), \dots, h_n(x)) \longmapsto h_i(x)$$

Linearity is straightforward. We need to check that $\bar{v}$ behaves well with respect to a change of coordinates.



Claim: $(k \circ h^{-1})' \begin{pmatrix} v(h_1) \\ \vdots \\ v(h_n) \end{pmatrix} = \begin{pmatrix} v(k_1) \\ \vdots \\ v(k_n) \end{pmatrix}$

Pf/ Write $k \circ h^{-1} =: w = (w_1, \ldots, w_n)$ with $w_i : \mathbb{R}^n \longrightarrow \mathbb{R}$.

For convenience, assume $h(p) = k(p) = 0$. Then

$$w_i(x) = w_i(x) - w_i(0) = \int_0^1 \frac{d}{dt} w_i(tx)\,dt =$$

$$= \int_0^1 \frac{d}{dt} w_i(tx_1, \ldots, tx_n)\,dt$$

$$= \int_0^1 \sum_{j=1}^n x_j \frac{\partial w_i}{\partial x_j}(tx)\,dt$$

$$= \sum_{j=1}^n x_j \int_0^1 \frac{\partial w_i}{\partial x_j}(tx)\,dt$$

$$= \sum_{j=1}^n x_j w_{ij}(x)$$

where $w_{ij}(x) := \int_0^1 \frac{\partial w_i}{\partial x_j}(tx)\,dt$, a C^∞ function.

Now, $k(x) = (k \circ h^{-1}) \circ h(x) = w \circ h(x) = (w_1 \circ h(x), \dots, w_n \circ h(x))$

$$= \left(\sum_{j=1}^{n} h_j(x) w_{1j}(h(x)), \dots, \sum_{j=1}^{n} h_j(x) w_{nj}(h(x)) \right)$$

i.e.
$$k_i(x) = \sum_{j=1}^{n} h_j(x) w_{ij}(h(x)),$$

Since v is a derivation,

$$v(k_i) = \sum_j \left[v(h_j) w_{ij}(\overbrace{h(p)}^{0}) + \overbrace{h_j(p)}^{0} v(w_{ij} \circ h) \right]$$

$$= \sum_j v(h_j) w_{ij}(0). \qquad (*)$$

Now, $w_i = \sum_j x_j w_{ij} \Rightarrow \frac{\partial w_i}{\partial x_l}(0) = \sum_j \frac{\partial x_j}{\partial x_l}\Big|_{x=0} w_{ij}(0) + \sum_j x_j\Big|_{x=0} \frac{\partial w_{ij}}{\partial x_l}(0)$

$$= w_{il}(0).$$

By $(*)$ we get $v(k_i) = \sum_j v(h_j) \frac{\partial w_i}{\partial x_j}(0)$.

In matrix form:

$$\underbrace{\begin{bmatrix} & \vdots & \\ \frac{\partial w_i}{\partial x_1}(0) & \cdots & \frac{\partial w_i}{\partial x_n}(0) \\ & \vdots & \end{bmatrix}}_{J(\underbrace{k \circ h^{-1}}_{w})} \begin{bmatrix} v(h_1) \\ \vdots \\ v(h_n) \end{bmatrix} = \begin{bmatrix} v(k_1) \\ \vdots \\ v(k_n) \end{bmatrix},$$

as required. $\square$

$$T_p^{phys}(M) \longrightarrow T_p^{geom}(M)$$

$$\begin{aligned} v : D_p(M) &\to \mathbb{R}^n \\ (U,h) &\mapsto v(U,h) \end{aligned} \longmapsto [\alpha_v], \quad \alpha_v : (-\varepsilon, \varepsilon) \longrightarrow M$$

To define α_v, pick a curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \Omega \subseteq h(U) \subseteq \mathbb{R}^n$.

with $\gamma(0) = h(p)$ and $\gamma'(0) = v(U,h)$. To be specific, say $\gamma(t) = h(p) + t\, v(U,h)$. Then let $\alpha_v = h^{-1} \circ \gamma : (-\varepsilon, \varepsilon) \longrightarrow M$.

The main thing to check is that $[\alpha_v]$ is independent of the choice of chart. So let (V,k) be another chart at p and define

$$\beta(t) = k^{-1}\big(k(p) + t\, v(V,k)\big).$$

We have $(k \circ \beta)'(0) = v(V,k)$ and

$$(k \circ \alpha)'(0) = \left[k \circ h^{-1}\left(h(p) + t\, v(U,h)\right)\right]'(0)$$

$$= (k \circ h^{-1})'(h(p))\, v(U,h) = v(V,k),$$

by definition of $T_p^{phys}(M)$. $\square$

Summary

curves $\qquad$ derivations $\qquad$ vectors / change of coords.

$$T_p^{\text{geom}}(M) \longrightarrow T_p^{\text{alg}}(M) \longrightarrow T_p^{\text{phys}}(M) \longrightarrow T_p^{\text{geom}}(M)$$

$$\alpha : (-\varepsilon, \varepsilon) \to M \;\longmapsto\; (f : M \to \mathbb{R}) \mapsto (f \circ \alpha)'(0)$$

$$v \;\longmapsto\; (U, h) \mapsto (v(h_1), \ldots, v(h_n))$$

$$v \;\longmapsto\; t \mapsto h^{-1}(h(p) + t\,v(U, h))$$