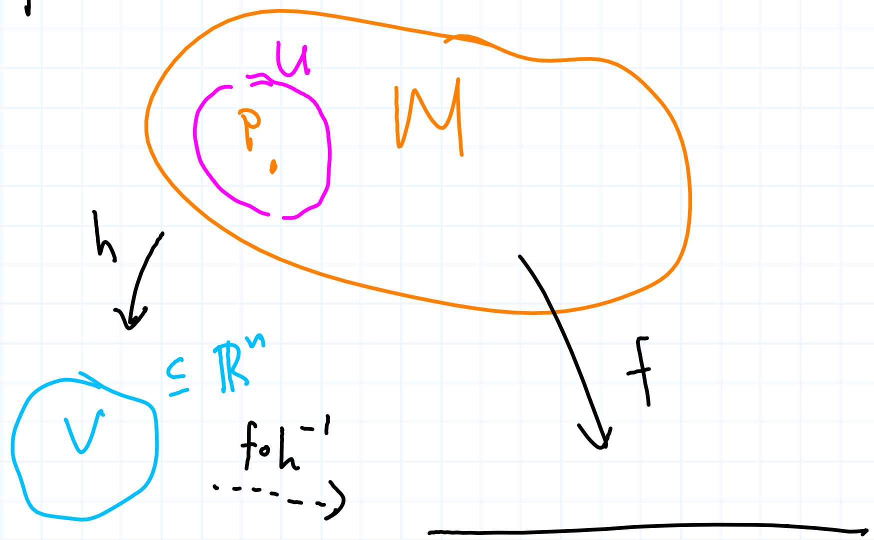
Math 411 1/29/10

Mappings

Let M be a manifold.

Def. $f: M \to \mathbb{R}$ is *differentiable at $p \in M$* if $f \circ h^{-1}$ is differentiable for some chart (U, h) at p.



HW. If $f: M \to \mathbb{R}$ is diff'ble at p with respect to (U, h) and if (V, k) is any chart at p, then f is diff'ble with respect to (V, k), too.
(You may use the fact that the composition of C^∞ functions is C^∞.)

Def. A continuous map $f: M \to N$ between manifolds is **differentiable at $p \in M$** if it is differentiable with respect to charts at $p \in M$ and $f(p) \in N$.

In detail:

(U, h) : chart at p ;

(V, k) : chart at $f(p)$

$$\begin{tikzcd}
M \arrow[r, "f"] & N \\
p \in U \arrow[u, phantom, "\subseteq" sloped] \arrow[d, "h"'] & V \ni f(p) \arrow[u, phantom, "\subseteq" sloped] \arrow[d, "k"] \\
\mathbb{R}^n \supseteq h(U) \arrow[r, dashed, "k \circ f \circ h^{-1}"] & k(V) \subseteq \mathbb{R}^m
\end{tikzcd}$$

Need $k \circ f \circ h^{-1}$ to be differentiable

Remarks.

1. $f: M \to N$ continuous $\Longrightarrow$ if (V, k) is a chart at $f(p)$, then $f^{-1}(V)$ is open in M and hence contains an open set U such that (U, h) is a chart.

2. $f: M \to N$ differentiable at p with respect to one choice of charts $\not\Rightarrow$ f differentiable with respect to all choices of charts.

Example
$$v_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$
$$(x,y,z) \longmapsto (x^2, xy, xz, y^2, yz, z^2)$$

Note: v_2 is well-defined:
$v_2(\lambda(x,y,z)) = \lambda^2 v_2(x,y,z) = v_2(x,y,z)$
for $\lambda \neq 0$.

Then v_2 is differentiable at all points of $\mathbb{P}^2$. For instance, let $p = (1, y, z) \in \mathbb{P}^2$ be an arbitrary point in U_x.
Let $x_0, \dots, x_5$ be the homogeneous coordinates of $\mathbb{P}^5$.
Then $v_2(p) \in U_{x_0} \subseteq \mathbb{P}^5$. Using the standard charts (U_x, φ_x) and (U_{x_0}, φ_{x_0}), we have

$$\varphi_y \circ v_2 \circ \varphi_x^{-1}(u,v) = \varphi_y(v_2(1,u,v)) = \varphi_y(1,u,v,u^2,uv,v^2) = (u,v,u^2,uv,v^2),$$

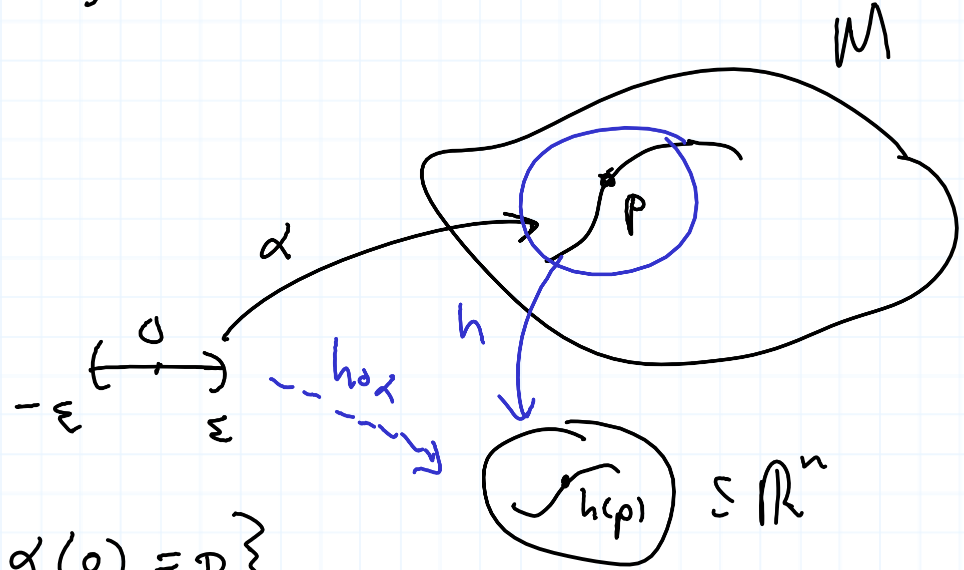
which is differentiable.

Def. A map $f: M \to N$ between manifolds is a *diffeomorphism* if f is differentiable and bijective with differentiable inverse.

Thm. (Milnor) $\exists$ two diff'ble structures $\mathcal{U}$ and $\mathcal{U}'$ on S^7 with no diffeomorphism $(S^7, \mathcal{U}) \longrightarrow (S^7, \mathcal{U}')$.

Tangent Space

M manifold



Version 1 (Geometric)

$$K_p = \{ \alpha : (-\varepsilon, \varepsilon) \xrightarrow{C^\infty} M : \varepsilon > 0,\ \alpha(0) = p \}$$

Def. For $\alpha, \beta \in K_p$, say $\alpha \sim \beta$ if $(h \circ \alpha)'(0) = (h \circ \beta)'(0)$ for some chart h at p.

An equivalence class $[\alpha]$ for $\sim$ is a tangent vector at p.

$$T_p^{geom}(M) = K_p(M)/\sim \; .$$

Version 2 (Algebraic)

$T_p^{alg}(M)$ = derivations of C^∞ germs of functions at p.

Def. Let f, g be C^∞ (real-valued) functions defined in neighborhoods about p. If $f = g$ on some neighborhood of p, say $f \sim g$. An equivalence class $[f]$ for $\sim$ is called a C^∞ germ of a function at p.

Let $\mathcal{E}_p$ denote the set C^∞ germs at p. Then $\mathcal{E}_p$ is naturally an algebra over $\mathbb{R}$.

A vector space / $\mathbb{R}$ which is also a ring. (We can scale, add, and multiply germs.)

A derivation of $\mathcal{E}_p$ is a linear map

$$v : \mathcal{E}_p \longrightarrow \mathbb{R}$$

obeying the product rule:

$$v(fg) = v(f)g(p) + f(p)v(g) .$$