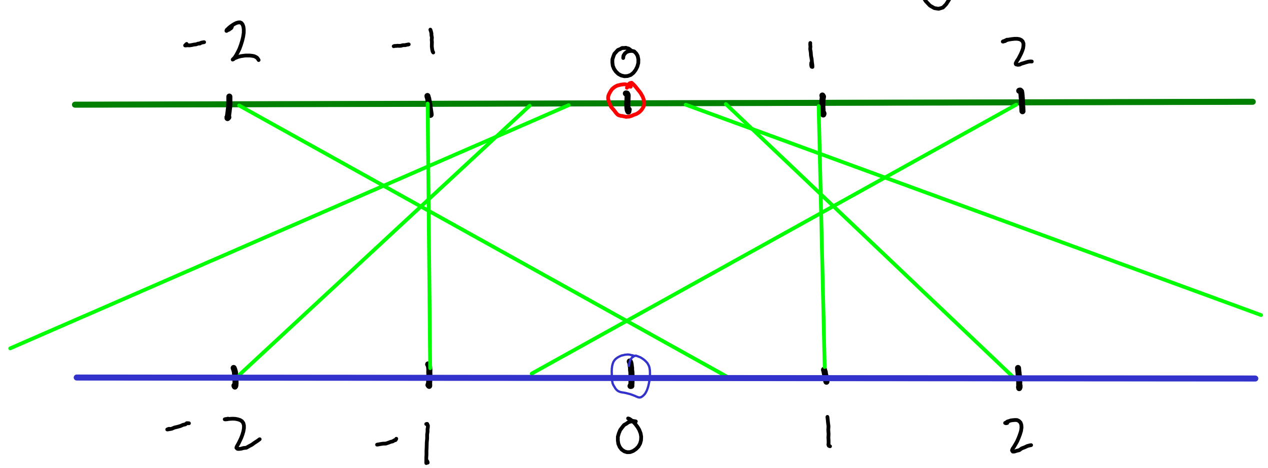
Math 411

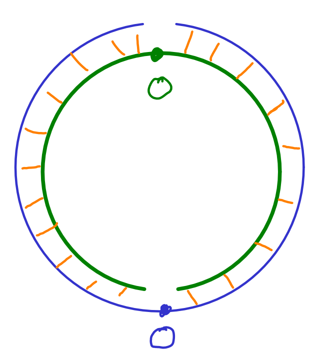
HW: ① Read J.P. May topology handout, §§ 1,2. ② Read our text §§ 1.1, 1.2

Manifold ≈ smooth blob on which one may do calculus.

Manifolds are formed by gluing together open subsets of $\mathbb{R}^n$.

Example Glue $\mathbb{R}$ to $\mathbb{R}$ using the function $\varphi(x) = \frac{1}{x}$.

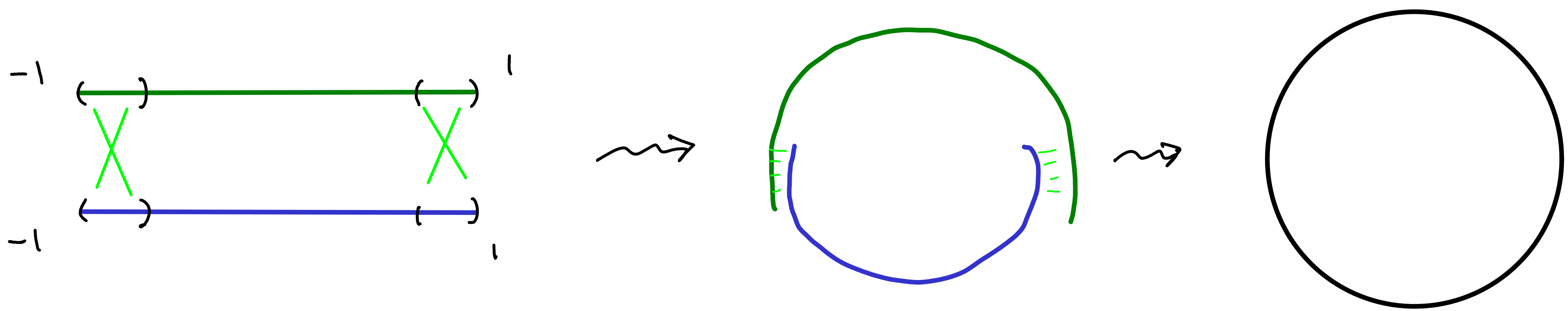




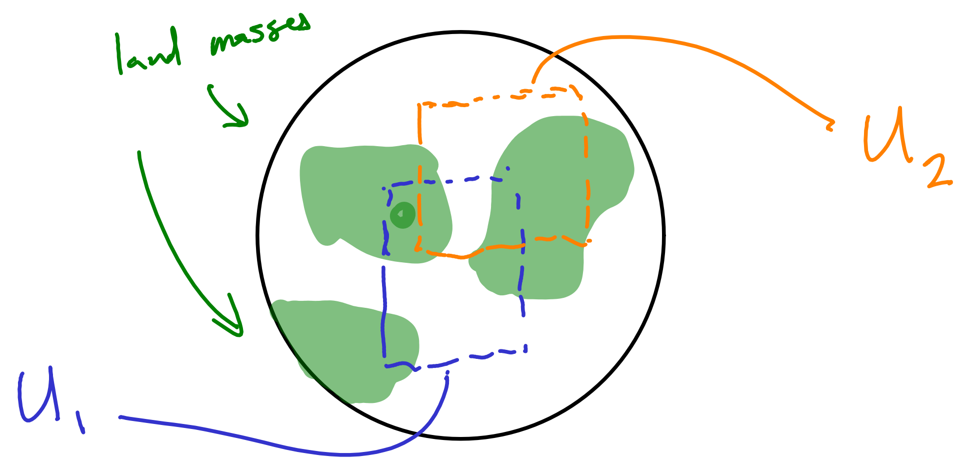
$= S^1 =$ circle

Example Another realization of S^1 as a manifold:

Glue $(-1,1)$ to $(-1,1)$ as indicated:



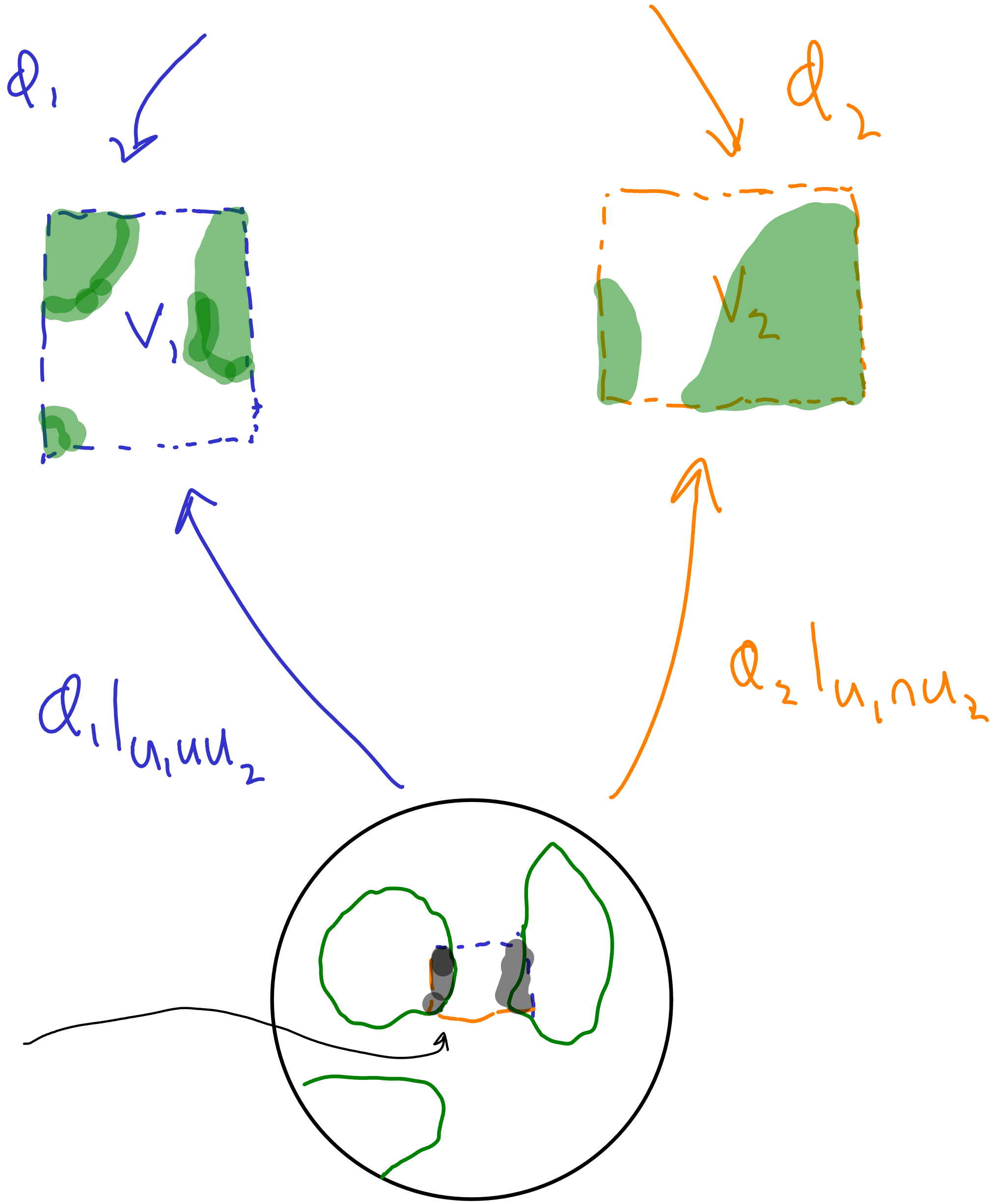
Example S^2 = sphere (globe)



Charts

$\varphi_1 : U_1 \to V_1 \subseteq \mathbb{R}^2$

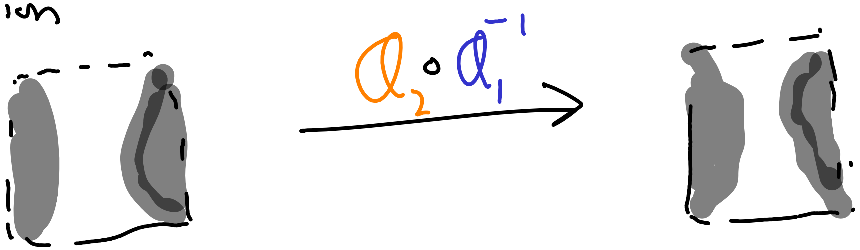
$\varphi_2 : U_2 \to V_2 \subseteq \mathbb{R}^2$



overlap

$U_1 \cap U_2$

Transition function



$$\begin{array}{ccc} \varphi_1(U_1 \cap U_2) & \xrightarrow{\varphi_2 \circ \varphi_1^{-1}} & \varphi_2(U_1 \cap U_2) \\ \cap & & \cap \\ \mathbb{R}^2 & & \mathbb{R}^2 \end{array}$$

We require that the transition function $\varphi_2 \circ \varphi_1^{-1}$ be smooth, i.e. all partial derivatives of all orders exist.

<u>Atlas</u> : A collection of charts covering the whole globe gives an atlas for the manifold.

Example: Projective Space

Def. $\mathbb{P}^2$ = lines in $\mathbb{R}^3$ passing through the origin

= 1-dimensional subspaces of $\mathbb{R}^3$.

We denote a point in $\mathbb{P}^2$ via its homogeneous coordinates:

If $\ell \in \mathbb{P}^2$, pick any non-zero point $(x,y,z) \in \ell$, and denote ℓ by just (x,y,z). Note that if $\lambda \in \mathbb{R} \setminus \{0\}$, then $\lambda(x,y,z)$ would work just as well. Hence, $\mathbb{P}^2 = \mathbb{R}^3 \setminus \{(0,0,0)\} / \sim$ where $\sim$ is the equivalence relation $(x,y,z) \sim \lambda(x,y,z)$ for all $\lambda \neq 0$.

Atlas Let $U_x = \{(x,y,z) \in \mathbb{P}^2 : x \neq 0\}$.

Note that this is well-defined: $x \neq 0$ iff $\lambda x \neq 0$ for all $\lambda \neq 0$.

$$\varphi_x : U_x \longrightarrow \mathbb{R}^2$$

$$(x,y,z) \longmapsto \left(\tfrac{y}{x}, \tfrac{z}{x}\right)$$

Let $U_y = \{ (x,y,z) \in \mathbb{P}^2 : y \neq 0 \}$, $\quad \phi_y : U_y \to \mathbb{R}^2$
$(x,y,z) \mapsto \left(\frac{x}{y}, \frac{z}{y}\right)$

$U_z = \{ (x,y,z) \in \mathbb{P}^2 : z \neq 0 \}$, $\quad \phi_z : U_z \to \mathbb{R}^2$
$(x,y,z) \mapsto \left(\frac{x}{z}, \frac{y}{z}\right)$.

Notes.

1. ϕ_x, ϕ_y, ϕ_z are bijective. For instance, the inverse of ϕ_x is $(a,b) \mapsto (1,a,b)$.
2. $\{U_x, U_y, U_z\}$ covers $\mathbb{P}^2$, i.e. $U_x \cup U_y \cup U_z = \mathbb{P}^2$