HW 11, due Friday, April 26 *Math 411*

1. A *linear subspace* L *of* $\mathbb{P}^n$ *of dimension* r, also called an r*-plane*, is an $(r+1)$-dimensional vector subspace $\tilde{L} \subseteq \mathbb{A}^{n+1}$ modulo the equivalence $p \sim \lambda p$ for every $p \in \tilde{L}$ and nonzero $\lambda \in k$. Given two linear subspaces L and M of $\mathbb{P}^n$, define their *span* to be the linear subspace determined by $\tilde{L} + \tilde{M}$ in $\mathbb{A}^{n+1}$. In other words, a point of $\mathrm{Span}\{L, M\} \subseteq \mathbb{P}^n$ is any nonzero $p+q$ with $p \in \tilde{L}$ and $q \in \tilde{M}$, modulo scaling. The intersection of linear spaces in $\mathbb{P}^n$ is a linear space, possibly empty (corresponding to the linear subspace $\{0\} \subset \mathbb{A}^{n+1}$). Note that $\dim L = \dim \tilde{L} - 1$.

   (a) Show that
   $$\dim \mathrm{Span}\{L, M\} = \dim L + \dim M - \dim(L \cap M)$$
   where the dimension of the empty set is taken to be 0. (Please use a nice short exact sequence to prove this.)

   (b) Show that $\dim(L \cap M) \geq \dim L + \dim M - n$. In particular, $L \cap M \neq \emptyset$ if $\dim L + \dim M \geq n$.

   (c) Thus, two planes, i.e., two 2-planes, in $\mathbb{P}^4$ must meet in a linear space of dimension at least 0, i.e., at least in a point. Give explicit examples showing that two planes in $\mathbb{P}^4$ can meet in a point, a line, or a plane. (There is more room in 4-space than in 3-space, where two planes must meet in at least a line.)

2. For each flag of linear spaces
   $$A_0 \subsetneq \cdots \subsetneq A_r$$
   we defined the Schubert variety
   $$\mathfrak{S}(A_0, \ldots, A_r) = \{L \in \mathbb{G}_r\mathbb{P}^n : \dim(L \cap A_i) \geq i \text{ for all } i\}$$
   and the corresponding Schubert class $(a_0, \ldots, a_r) \in A^*(\mathbb{G}_r\mathbb{P}^n)$ where $a_i = \dim A_i$. (Working modulo rational equivalence of cycles, only the dimensions of the A_i matter.) Each Schubert class represents r-planes in $\mathbb{G}_r\mathbb{P}^n$ satisfying certain conditions. For instance, in $A^*(\mathbb{G}_1\mathbb{P}^3)$, the class $(0, 2)$ is the class of a Schubert variety $\mathfrak{S}(A_0, A_1)$ where A_0 is a point and A_1 is a plane. This Schubert variety consists of lines L passing through the point A_0 and lying in the plane A_1 (in order to satisfy the conditions $\dim(L \cap A_i) \geq i$).

   (a) Let $n = 3$. For $r = 0, 1, 2, 3$, describe all possible Schubert classes $(a_0, \ldots, a_r)$ and state the corresponding condition placed on r-planes in $\mathbb{P}^3$. (Note that $0 \leq a_0 < \cdots < a_r \leq 3$.)

(b) The class $(a_0, \ldots, a_r)$ is an element of $A^\ell(\mathbb{G}_r\mathbb{P}^n)$ where the codimension ℓ is given by

$$\ell = (r+1)(n-r) - \sum_{i=0}^{r}(a_i - i).$$

Calculate the codimension for each of the classes in part (a).

3. The set of lines in $\mathbb{P}^3$ meeting a given line L is a hyperplane section of $\mathbb{G}_1\mathbb{P}^3 \subset \mathbb{P}^5$, where as usual we consider the Grassmannian embedded via the Plücker embedding. In other words, there is a hyperplane $H_L \subset \mathbb{P}^5$ such that the lines meeting L are given by $\mathbb{G}_1\mathbb{P}^3 \cap H_L$. The collection of lines meeting 4 given lines $L_1, \ldots, L_4$ is then

$$\cap_i(\mathbb{G}_1\mathbb{P}^3 \cap H_{L_i}) = \mathbb{G}_1\mathbb{P}^3 \cap (\cap_i H_{L_i}).$$

If the L_i are general lines, then $\cap_i H_{L_i}$ will be a line in $\mathbb{P}^5$, which we expect to meet the quadric hypersurface $\mathbb{G}_1\mathbb{P}^3 \subset \mathbb{P}^5$ in two points. The point of this exercise is to compute an explicit example.

The first step is to get explicit equations for the hyperplanes described above. A line L in $\mathbb{P}^3$ can be parametrized by $L(s,t) = sp + tq$ where $p, q \in \mathbb{P}^3$ are any two fixed distinct points on the line and (s,t) varies over $\mathbb{P}^1$. Consider the matrix

$$C = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}.$$

The first two rows give the homogeneous coordinates of L as an element of the Grassmannian: their span is L. The last two rows span an arbitrary line, M, in $\mathbb{P}^3$. The lines L and M intersect in $\mathbb{P}^3$ if and only if their corresponding 2-dimensional subspaces in $\mathbb{A}^4$ meet in a subspace of at least affine dimension 1. In other words, if and only if C has a non-trivial kernel. So the condition that $L \cap M \neq \emptyset$ is equivalent to $\det C = 0$. To get the equation of the hyperplane, calculate $\det C$ using a generalized Laplace expansion along the first two rows of C.

Generalized Laplace Expansion. Let D be an $n \times n$ matrix. Let $[n] = \{1, \ldots, n\}$, and fix row indices $I \subset [n]$. The complement is denoted $I^c = [n] \setminus I$. For each collection of column indices J having the same number of elements as I, i.e., $|I| = |J|$, we write $D_{I,J}$ for the corresponding submatrix of D. Then

$$\det D = \sum_{J \subset [n], |J|=|I|} (-1)^{\sum_{i\in I} i + \sum_{j \in J} j} \det D_{I,J} \det D_{I^c,J^c}.$$

Expanding $\det C$ along the first two rows gives

$$\begin{aligned} \det C \;=\; & (p_0q_1 - p_1q_0)(x_2y_3 - x_3y_2) - (p_0q_2 - p_2q_0)(x_1y_3 - x_3y_1) \\ & + (p_0q_3 - p_3q_0)(x_1y_2 - x_2y_1) + (p_1q_2 - p_2q_1)(x_0y_3 - x_3y_0) \\ & - (p_1q_3 - p_3q_1)(x_0y_2 - x_2y_0) + (p_2q_3 - p_3q_2)(x_0y_1 - x_1y_0). \end{aligned}$$

Note that the Plücker coordinates for the line spanned by x and y are

$$(x_0y_1 - x_1y_0, x_0y_2 - x_2y_0, x_0y_3 - x_3y_0, x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2).$$

Let $z_{01}, z_{02}, z_{03}, z_{12}, z_{13}, z_{23}$ be the coordinates on $\mathbb{P}^5$, and let

$$[ij] = \det \begin{pmatrix} p_i & p_j \\ q_i & q_j \end{pmatrix}.$$

We see that $\det C = 0$ defines the intersection of $\mathbb{G}_i\mathbb{P}^3 \subset \mathbb{P}^5$ with the hyperplane

$$H_L = [23]z_{01} - [13]z_{02} + [12]z_{03} + [03]z_{12} - [02]z_{13} + [01]z_{23} = 0.$$

Let (u, x, y, z) be the coordinates on $\mathbb{P}^3$, and consider the four lines given in homogeneous equations by

$$\begin{aligned} L_1 &= \{y = z = 0\}, & L_2 &= \{x = z, y = u\} \\ L_3 &= \{x = 2z, y = 2u\}, & L_4 &= \{x = y, z = u\}. \end{aligned}$$

(a) Compute the four hyperplanes, H_{L_i}.

(b) Compute a parametric equation for the line $\cap_{i=1}^4 H_{L_i}$.

(c) Find the two points of intersection of that line with the Grassmannian.

(d) Describe these two points as lines in $\mathbb{P}^3$.