HW 10, due Friday, April 19 *Math 411*

1. Let P be the hexagon in the plane with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(2,1)$, $(1,2)$, and $(2,2)$.
   (a) Draw the fan, Δ, for the corresponding toric variety, X, labeling the first lattice points along its 1-dimensional cones.
   (b) Choose two adjacent 2-dimensional cones in Δ, construct the two corresponding affine toric varieties, and show the gluing instructions.
   (c) Show that X smooth.
   (d) Calculate the Chow ring, $A^\bullet(X)$.
   (e) Calculate the cohomology $H^k X$ for all k.
   (f) Give the mapping of X into $\mathbb{P}^6$ determined by the 7 lattice points of P (there is one interior point) in homogeneous coordinates.
   (g) Pick two components of your mapping, and show that they have the same degree.
   (h) Describe X as a quotient, $(\mathbb{C}^{\Delta(1)} \setminus Z)/(x \sim \ell \cdot x)$ according to Cox's theorem.

2. We have seen that the toric variety Y determined by the single cone, $\sigma = \mathbb{R}_{>0}(2,-1) + \mathbb{R}_{>0}(0,1)$, has semigroup algebra

$$\mathbb{C}[x, xy, xy^2] \approx \mathbb{C}[u,v,w]/(uw - v^2).$$

   Thus, $Y = \{(u,v,w) \in \mathbb{C}^3 : uw = v^2\}$, a cone. The fact that Y has a singularity can be inferred from the cone since

$$\det \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} = -2 \neq \pm 1.$$

   To desingularize Y we "blow-up" the singular point by splitting σ into two "nonsingular" cones. Let Δ be the fan with maximal cones

$$\begin{aligned} \sigma_1 &= \mathbb{R}_{>0}(2,-1) + \mathbb{R}_{>0}(1,0) \\ \sigma_2 &= \mathbb{R}_{>0}(1,0) + \mathbb{R}_{>0}(0,1), \end{aligned}$$

   and let $X = X(\Delta)$ be the corresponding toric variety.
   (a) Describe how X is obtained from gluing two copies of $\mathbb{C}^2$.
   (b) Describe a mapping $\pi \colon X \to Y$ such that $\pi^{-1}(p)$ consists of a single point for all $p \in Y \setminus \{(0,0,0)\}$ and such that $\pi^{-1}(0,0,0)$ is a "line". (Letting x and y be the indeterminates corresponding to the lattice points $(1,0)$ and $(0,1)$, respectively, and writing all coordinate functions in terms of x and y should guide the way. Show how the mapping is defined on the two copies of $\mathbb{C}^2$ that are glued to form X.)

3. (a) Suppose $L\colon V \to W$ and $M\colon V' \to W'$ are linear mappings of finite-dimensional vector spaces. There is an induced mapping

$$\begin{aligned} L \otimes M\colon V \otimes V' &\to W \otimes W' \\ v \otimes v' &\mapsto L(v) \otimes M(v') \end{aligned}$$

Choosing bases $v_1, \ldots, v_n$ for V; $w_1, \ldots, w_m$ for W; $v'_1, \ldots, v'_t$ for V'; and $w'_1, \ldots, w'_s$ for W', we identify L and M with matrices. Choosing the corresponding bases

$$v_1 \otimes v'_1, v_1 \otimes v'_2, \ldots, v_1 \otimes v'_t, v_2 \otimes v'_1, \ldots, \ldots, v_n \otimes v'_t$$

for $V \otimes V'$ and

$$w_1 \otimes w'_1, w_1 \otimes w'_2, \ldots, w_1 \otimes w'_s, w_2 \otimes w'_1, \ldots, \ldots, w_m \otimes w'_s$$

for $W \otimes W'$ (in the given orders) describe $L \otimes M$ as a matrix.

(b) With the above conventions, what is $L \otimes M$ for

$$L = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

4. Consider an exact sequence of finite-dimensional vectors spaces:

$$0 \to V_1 \to V_2 \to \cdots \to V_k \to 0.$$

Prove that $\sum_{i=1}^{k} (-1)^i \dim V_i = 0$.