HW 9, due Friday, April 12 *Math 411*

1. Let V be an oriented vector space with scalar product $\langle\ ,\ \rangle$ of index s. Let $e_1, \ldots, e_n$ be an orthonormal basis for V with $\langle e_i, e_i\rangle = \varepsilon_i \in \{-1, 1\}$ for each i, and let ω_V be the volume form for V. For each k, let $*\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ be the $*$-operator.

   (a) Compute $*1$ and $*\omega_V$.

   (b) Show $** = (-1)^{k(n-k)+s}\,\mathrm{Id}_{\Lambda^k V^*}$. (Recall that if $e_1, \ldots, e_n$ is a positively oriented orthonormal basis for V, then we have an orthonormal basis $\{e_\mu^*\}$ for $\Lambda^k V^*$ with $\langle e_\mu^*, e_\gamma^*\rangle = \delta_{\mu\gamma}\varepsilon_\mu$ where $\varepsilon_\mu = \prod_i \varepsilon_{\mu_i}$. We also have $\mathrm{sgn}(\tau_\mu)$ defined by

   $$e_\mu^* \wedge e_{\tilde{\mu}}^* = \mathrm{sgn}(\tau_\mu)\,\omega_V = \mathrm{sgn}(\tau_\mu)\, e_1^* \wedge \cdots \wedge e_n^*,$$

   where $\tilde{\mu}$ is the complement of μ in $\{1, \ldots, n\}$. Then $*e_\mu^* = \varepsilon_\mu\,\mathrm{sgn}(\tau_\mu)\, e_{\tilde{\mu}}^*$.)

   (c) Show $\langle *\eta, *\zeta\rangle = (-1)^s\langle \eta, \zeta\rangle$ for all $\eta, \zeta \in \Lambda^k V^*$. (Try to do this without appealing to the description of the $*$-operator in terms of a basis. Use defining properties of the $*$-operator, instead.)

   (d) For $\eta, \zeta \in \Lambda^k V^*$, find the relationship between (i) $*(\eta \wedge *\zeta)$, (ii) $*(*\eta \wedge \zeta)$, (iii) $*(\zeta \wedge *\eta)$, and (iv) $*(*\zeta \wedge \eta)$ by relating them all to $\langle \eta, \zeta\rangle$.

2. Let $V = \mathbb{R}^4$ with standard oriented basis $e_1, \ldots, e_4$ and with the Lorentzian scalar product:

   $$\langle e_i, e_j\rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \text{ and } i \neq 4, \\ -1 & \text{if } i = j = 4. \end{cases}$$

   For each $k = 0, 1, 2, 3, 4$, describe the $*$-operator by showing what it does to the standard basis for $\Lambda^k V^*$. (For instance,

   $$e_1^* \wedge e_2^* \wedge e_3^*, \quad e_1^* \wedge e_2^* \wedge e_4^*, \quad e_1^* \wedge e_3^* \wedge e_4^*, \quad e_2^* \wedge e_3^* \wedge e_4^*$$

   is the standard basis for $\Lambda^3 V^*$. Order the indices lexicographically, as above.)

3. Now let $V = \mathbb{R}^3$ with the standard inner product and orientation. What is the precise relation between the $*$-operator and the usual cross product (with proof)?

4. What are the cohomology groups of a solid donut?

5. What are the cohomology groups of (the surface of) a donut with g holes? (Recall that in the last homework, you computed the cohomology of a punctured single-holed donut.) You may use the fact that if M is a compact, connected, orientable n-manifold with $n \geq 2$, and $p \in M$, then $H^n(M \setminus \{p\}) = 0$.

6. Consider the cone, σ, in $\mathbb{R}^2$ generated by the vectors $(0, 1)$ and $(3, -1)$. Describe the (affine) toric variety, U_σ.

7. Each of the quadrants of $\mathbb{R}^2$ forms a cone. Consider the fan, Δ, consisting of these four cones. Describe the toric variety $X(\Delta)$ by identifying the affine toric varieties corresponding to each cone and describing the gluing instructions. Extra: Can you identify $X(\Delta)$ as a well-known manifold?