1. Manifolds M and N are *homotopy equivalent* if there are maps $f\colon M \to N$ and $g\colon N \to M$ such that $g \circ f \sim \mathrm{id}_M$ and $f \circ g \sim \mathrm{id}_N$ (where $\sim$ denote homotopy equivalence of maps). Show that if M and N are homotopy equivalent then $H^k M \approx H^k N$ for $k \geq 0$.

2. Let X be a topological space, and let $A \subseteq X$ be a subspace. A *retraction* is a continuous mapping
$$r\colon X \to A$$
such that $r(a) = a$ for all $a \in A$. A *deformation retraction* is a homotopy between the identity and a retraction, that is, a continuous mapping
$$h\colon [0,1] \times X \to X$$
such that $h_0(x) := h(0,x) = x$ for all $x \in X$, while $h_1(x) := h(1,x) \in A$, and $h_1(a) = a$ for all $x \in X$ and for all $a \in A$. (So, more accurately, h is a homotopy between (1) the identity, and (2) a retraction composed with the inclusion of A into X.) In this case, A is called a *deformation retract* of X.

   In the category of manifolds, we have the same definitions, but all mappings are required to be smooth.

   (a) Show that if N is a deformation retract of the manifold M, then $H^k(M) \approx H^k(N)$ for $k \geq 0$.

   (b) Show that S^n is a deformation retract of the punctured plane, $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$. (Hence, these two manifolds have the same cohomology.)

3. In class, we showed that if M is a closed orientable n-manifold without boundary, then $H^n M \neq 0$. Show that the result does not hold without the compactness assumption. Where is compactness used in the proof we gave in class?

4. Use Mayer-Vietoris to compute the cohomology of the following manifolds:

   (a) The punctured plane, $\mathbb{R}^2 \setminus \{(0,0)\}$. Here, take $U = \mathbb{R}^2 \setminus (x\text{-axis})$ and take $V = \mathbb{R}^2 \setminus (y\text{-axis})$.

   (b) The twice punctured plane, $\mathbb{R}^2 \setminus \{(-1,0),(1,0)\}$.

   (c) The 2-torus, $T = S^1 \times S^1$. The only way I could think of doing this was as follows: Let $p \in T$, let U be a small open disk on T containing p, and let $V = T \setminus \{p\}$. It turns out that V is diffeomorphic to a sort of figure-eight band (most easily seen by drawing the torus as a square with sides identified in the usual way, and letting p be the center of the square). I then used Mayer-Vietoris to compute the cohomology of V, and plugged in the result for the Mayer-Vietoris sequence for T. The sequence is still ambiguous, but problem 3, above, fixes that.

5. Two technical results used in class:

   (a) Let $e_1, \ldots, e_n$ and $v_1, \ldots, v_n$ be two ordered bases for a vector space V, and let $e_1^*, \ldots, e_n^*$ and $v_1^*, \ldots, v_n^*$. Say $v_j = \sum_i a_{ij} e_i$ and $v_j^* = \sum_i b_{ij} e_i^*$. Define matrices $A = (a_{ij})$ and $B = (b_{ij})$. Prove that $B = (A^t)^{-1}$.

   (b) Let V be as above, and suppose $\langle \ , \ \rangle$ is a bilinear form on V. Define the matrices $G = (\langle e_i, e_j \rangle)$ and $H = (\langle v_i, v_j \rangle)$, and let A be the matrix defined above. Show that
$$H = A^t G A.$$