HW 5, due Friday, March 8 *Math 411*

1. Let V and W be vector spaces over an arbitrary field K, and suppose that V has finite dimension n.

   (a) Show that
   $$\hom(V,W) \approx W^n = \underbrace{W \times \cdots \times W}_{n \text{ factors}}.$$

   (b) There is a mapping of vector spaces
   $$\begin{aligned} V^* \otimes W &\to \hom(V,W) \\ \phi \otimes w &\mapsto [v \mapsto \phi(v)w] \end{aligned}$$
   It's induced by the corresponding bilinear mapping from $V^* \times W$ via the universal property for tensor products. Show that this mapping is an isomorphism. [We use finite-dimensionality here. It is useful to pick a basis for V and its corresponding dual basis.]

   (c) The cross product
   $$\begin{aligned} \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{R}^3 \\ (u,v) &\mapsto u \times v \end{aligned}$$
   is an alternating, multilinear mapping. Hence, it factors through a unique mapping $\Lambda^2\mathbb{R}^3 \to \mathbb{R}^3$, i.e., to an element of $\hom(\Lambda^2\mathbb{R}^3, \mathbb{R}^3)$. By the previous exercise, we can identify this element with an element of $(\Lambda^2\mathbb{R}^3)^* \otimes \mathbb{R}^3$, and hence with an element of $\Lambda^2((\mathbb{R}^3)^*) \otimes \mathbb{R}^3$. Letting e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$, the space $\Lambda^2((\mathbb{R}^3)^*) \otimes \mathbb{R}^3$ has a basis $\{(e_i^* \wedge e_j^*) \otimes e_k\}$ where $1 \le i < j \le 3$ and $1 \le k \le 3$. Identify the cross product in terms of this basis.

2. Orientation.

   (a) Is the standard atlas for $\mathbb{P}^1$ an orienting atlas? If not, can you fix it?

   (b) Same question for $\mathbb{P}^3$.

3. Consider the differential form
   $$\eta = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy \in \Omega^2\mathbb{R}^3.$$
   Let $\iota\colon S^2 \to \mathbb{R}^3$ be the standard embedding of the 2-sphere, and let $\omega = \iota^*\eta \in \Omega^2 S^2$. Compute $\int_{S^2} \omega$ by choosing charts and using the definition of integration. Check your result with Stokes' theorem.

4. Let X be a topological space, and let Y be a subset of X. The *subspace topology* on Y is given by the collection of sets $\{Y \cap U : U \text{ open in } X\}$. Thus, a set in Y is open in the subspace topology if and only if it is the intersection of an open subset of X with Y. The term "open set" can be ambiguous in this context; to be clear, you can say "open in Y" or "open in X".

   Let $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following are open in Y, and which are open in $\mathbb{R}$?

   (a) $A = \{x : \frac{1}{2} < |x| < 1\}$.
   (b) $B = \{x : \frac{1}{2} < |x| \leq 1\}$.
   (c) $C = \{x : \frac{1}{2} \leq |x| < 1\}$.
   (d) $D = \{x : \frac{1}{2} \leq |x| \leq 1\}$.
   (e) $E = \{x : 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}_{>0}\}$.

5. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology with basis $U \times V$ where U and V are open sets in X and Y, respectively. Is it true that every open set in the product topology has the form $U \times V$ with U and V open in X and Y, respectively? Proof or counterexample.

6. A function $f : X \to Y$ between topological spaces is *continuous* if $f^{-1}(V)$ is open in X for each open set V in Y. The function f is *open* of $f(U)$ is open in Y for each open set U in X.

   (a) Let X and Y be topological spaces, and consider $X \times Y$ with the product topology. Let $\pi : X \times Y \to X$ be the first projection: $\pi(x, y) = x$.
      i. Show that π is continuous.
      ii. Show that π is open.

   (b) (Note: the following exercises would work just as well if one substituted the word "homeomorphism" for "diffeomorphism" everywhere. However, the result arose in the context of diffeomorphisms in class.) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open sets, and let $f : X \to Y$ be a local diffeomorphism. This means that for each $x \in X$ there is an open neighborhood $U_x \subset X$ of x such that $f(U_x)$ is open and $f|_{U_x}$ is a diffeomorphism onto $f(U_x)$.
      i. Give a simple example of a local diffeomorphism which is not a diffeomorphism.
      ii. Show that a local diffeomorphism is an open mapping.

7. Let V and W be vector spaces over a field k, and suppose $L : V \to W$ is a linear function. We have defined

$$L^* : \Lambda^\ell W^* \to \Lambda^\ell V^*$$
$$\phi_1 \wedge \cdots \wedge \phi_\ell \mapsto (\phi_1 \circ L) \wedge \cdots \wedge (\phi_\ell \circ L)$$

Show that for all $v_1, \dots, v_\ell \in V$,

$$L^*(\phi_1 \wedge \cdots \wedge \phi_\ell)(v_1, \dots, v_\ell) = (\phi_1 \wedge \dots \phi_\ell)(Lv_1, \dots, Lv_\ell).$$