HW 4, due Friday, March 1 *Math 411*

1. **Computations.**

   (a)

$$\begin{aligned} f\colon \mathbb{R}^2 &\to \mathbb{R}^4 \\ (x,y) &\mapsto (x^2, 2x+y, y^4, xy) \end{aligned}$$

   i. Let $\omega = y_1 dy_1 \wedge dy_2 + (y_1 y_3) dy_3 \wedge dy_4 \in \Omega^2\mathbb{R}^4$. Compute $f^*\omega$ and express your answer in terms of the standard basis for $\Omega^2\mathbb{R}^2$.

   ii. Consider the vector field $v = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $T\mathbb{R}^2$. Compute $f_{*,(1,1)}v$, i.e., $df_{(1,1)}(v)$, in terms of the standard basis for $T\mathbb{R}^4$.

   (b) Consider the polar coordinates map

$$\begin{aligned} f\colon I := (0,1)\times(0,2\pi) &\to \mathbb{R}^2 \\ (r,\theta) &\mapsto (r\,\cos\theta, r\,\sin\theta) \end{aligned}$$

   and the "volume form", $\omega := dx \wedge dy \in \Omega^2\mathbb{R}^2$. Compute $f^*\omega \in \Omega^2 I$.

2. Suppose that ω is a 1-form on $\mathbb{P}^1$. Let (U_x, ϕ_x) and (U_y, ϕ_y) denote the two standard charts for $\mathbb{P}^1$. So $\phi_x\colon U_x \xrightarrow{\sim} \mathbb{R}^1$ and similarly for ϕ_y. Say $f(a)\,da$ is ω in (U_x, ϕ_x) coordinates and $g(b)\,db$ is ω in (U_y, ϕ_y) coordinates. On the overlap, $U_x \cap U_y$ this gives representations for ω, so it make sense to compare them.

   (a) What is $f(a)\,da$ in terms of $g(b)\,db$? In other words, compute the pullback $(\phi_y \circ \phi_x^{-1})^*(g(b)\,db)$.

   (b) In light of your answer to part (a), construct a nonzero (globally defined) 1-form on $\mathbb{P}^1$.

3. Consider the 2-sphere with its usual embedding in space, $\iota\colon S^2 \to \mathbb{R}^3$. Let $\omega = x\,dx + y\,dy + z\,dz \in T^*\mathbb{R}^3$. What is $\iota^*\omega \in T^*S^2$?

   (a) Compute the pullback with respect to the charts for S^2 given in the last homework assignment.

   (b) Explain why your answer to (a) makes sense (i.e., could have been surmised without calculation).

4. **Another characterization of tangent space.** Let M be an n-dimensional manifold. For each $p \in M$, let ξ_p be the $\mathbb{R}$-algebra of germs of functions at p. Let $\mathfrak{m}_p \subset \xi_p$ denote the ideal of germs vanishing at p. (Recall that the value of $f \in \xi_p$ at p is well-defined;

so in particular, the notion of a germ being zero at p is well-defined). The purpose of this exercise is to show

$$\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^* \approx T_pM,$$

where $\mathfrak{m}_p^2$ is the square of the ideal $\mathfrak{m}_p$.

(a) Think of T_pM as the space of derivations of germs and define

$$\begin{aligned}\alpha\colon (\mathfrak{m}_p/\mathfrak{m}_p^2)^* &\to T_pM\\ \phi &\mapsto \alpha(\phi)\end{aligned}$$

where

$$\begin{aligned}\alpha(\phi)\colon \xi_p &\to \mathbb{R}\\ f &\mapsto \phi(f - f(p)).\end{aligned}$$

Linearity of both α and $\alpha(\phi)$ is straightforward (check it on your own). Prove that $\alpha(\phi)$ is a derivation. Hint:

$$fg - f(p)g(p) = (f - f(p))(g - g(p)) + f(p)(g - g(p)) + g(p)(f - f(p)).$$

(b) Now define

$$\begin{aligned}\beta\colon T_pM &\to (\mathfrak{m}_p/\mathfrak{m}_p^2)^*\\ v &\mapsto \beta(v)\end{aligned}$$

where

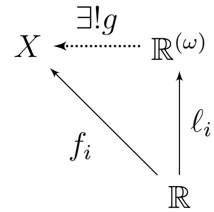
$$\begin{aligned}\beta(v)\colon \mathfrak{m}_p/\mathfrak{m}_p^2 &\to \mathbb{R}\\ f &\to v(f).\end{aligned}$$

i. Show that $\beta(v)$ is well-defined.

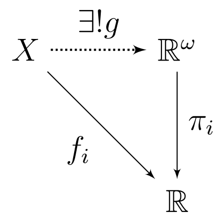
ii. Show that α and β are inverses.

5. Let $\mathbb{R}^{(\omega)} := \bigoplus_{i=1}^{\infty} \mathbb{R}$, the collection of all sequences of real numbers with only a finite number of nonzero terms. Let $\mathbb{R}^{\omega} := \prod_{i=1}^{\infty} \mathbb{R}$, the collection of all sequences of real numbers.

(a) Show that $\mathbb{R}^{(\omega)}$ and $\mathbb{R}^{\omega}$ are a categorical coproduct and product, respectively, in the category of vector spaces. For instance, first consider $\mathbb{R}^{(\omega)}$. For $i = 1, 2 \ldots,$ there are canonical injections $\ell_i\colon \mathbb{R} \to \mathbb{R}^{(\omega)}$ sending $x \in \mathbb{R}$ to the sequence whose i-th term is x and whose other terms are zeroes. Suppose X is a real vector space and you are given (linear) mappings $f_i\colon \mathbb{R} \to X$ for each i. Show there is a unique mapping g so that the following diagram commutes for each i:



The mapping g is usually denoted $\oplus_i f_i \colon \mathbb{R}^{(\omega)} \to X$. To show that $\mathbb{R}^\omega$ is the product, you need to show the "dual" result, turning all the arrows around. There are canonical projections $\pi_i \colon \mathbb{R}^\omega \to \mathbb{R}$ sending a sequence to its i-th term. Show that given mappings $f_i \colon X \to \mathbb{R}$ for each i, there exists a unique mapping g so that the following diagram commutes for each i:



(b) Show that $(\mathbb{R}^{(\omega)})^* \approx \mathbb{R}^\omega$. So here is an example of a vector space V for which V^* is not isomorphic to its dual. [It is impossible to have a linear isomorphism between $\mathbb{R}^{(\omega)}$ and $\mathbb{R}^\omega$ since only one has countable dimension.] Recall that $V^* \approx V$ whenever V is finite-dimensional.