Math 411

Let $f: M \to N$. Show that the ordinary derivative glues together to give a mapping $df := f' := f_* : TM \to TN$.

Solution/ Take charts (U,h) at $p \in M$ and (V,k) at $f(p) \in N$ with $U \subseteq f^{-1}(V)$. So $TM \supseteq \pi_M^{-1}(U) \simeq U \times \mathbb{R}^m \simeq h(U) \times \mathbb{R}^m$
$TN \supseteq \pi_N^{-1}(V) \simeq V \times \mathbb{R}^n \simeq k(V) \times \mathbb{R}^n$. Define

$$f_* : h(U) \times \mathbb{R}^m \longrightarrow k(V) \times \mathbb{R}^n$$
$$(x, v) \mapsto \left((kfh^{-1})(x), (kfh^{-1})'(x)\, v\right).$$

To show that f_*, here defined locally, glues together, take two pairs of charts (U_i, h_i), (V_i, k_i) with $U_i \subseteq f^{-1}(V_i)$, $i = 1, 2$, then note that we have a commutative diagram

$$\begin{tikzcd}
h_1(U_1 \cap U_2) \times \mathbb{R}^m \arrow[r, "{(k_1 f h_1^{-1},\, (k_1 f h_1^{-1})')}"] \arrow[d, "{(h_2 h_1^{-1},\, (h_2 h_1^{-1})')}"'] \arrow[dr, phantom, "///"] & k_1(V_1 \cap V_2) \times \mathbb{R}^n \arrow[d, "{(k_2 k_1^{-1},\, (k_2 k_1^{-1})')}"] \\
h_2(U_1 \cap U_2) \times \mathbb{R}^m \arrow[r, "{(k_2 f h_2^{-1},\, (k_2 f h_2^{-1})')}"'] & k_2(V_1 \cap V_2) \times \mathbb{R}^n
\end{tikzcd}$$

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