Math 374

Threshold density $\zeta_\tau(0000)$ on K_4

degree	divisor length
0	$\lambda(0000) := 1$
1	$\lambda(0001) = \lambda(0000) = 1$
2	$\lambda(0011) = 2\lambda(0001) = 2$
	$\lambda(0002) = \lambda(0001) = 1$
3	$\lambda(0111) = 3\lambda(0011) + \lambda(0002) = 3\cdot 2 + 1 = 7$
	$\lambda(0012) = \lambda(0011) + \lambda(0002) = 2 + 1 = 3$
4	$\lambda(1111) = 4\lambda(0111) = 4\cdot 7 = 28$
	$\lambda(0112) = \lambda(0111) + 3\lambda(0012) = 7 + 3\cdot 3 = 16$
	$\lambda(0022) = 2\lambda(0012) = 2\cdot 3 = 6$
5	$\lambda(1112) = 3\lambda(0112) + \lambda(1111) = 3\cdot 16 + 28 = 76$
	$\lambda(0122) = 2\lambda(0022) + 3\lambda(0112) = 2\cdot 6 + 3\cdot 16 = 60$

6 $\lambda(1122) = 2\lambda(1112) + 2\lambda(0122) = 2\cdot 76 + 2\cdot 60 = 272$

$\lambda(0222) = 3\lambda(0122) + \lambda(1112) = 3\cdot 60 + 76 = 256$

$\lambda(0123) = \lambda(0122) = 60$

7 $\lambda(1222) = 3\lambda(1122) + \lambda(0222) = 3\cdot 272 + 256 = 1072$

$\lambda(1123) = \lambda(1122) = 272$

$\lambda(0223) = \lambda(0222) = 256$

8 $\lambda(2222) = 4\lambda(1222) = 4\cdot 1072 = 4288$

$\lambda(1223) = \lambda(1222) = 1072$

9 $\lambda(2223) = \lambda(2222) = 4288$

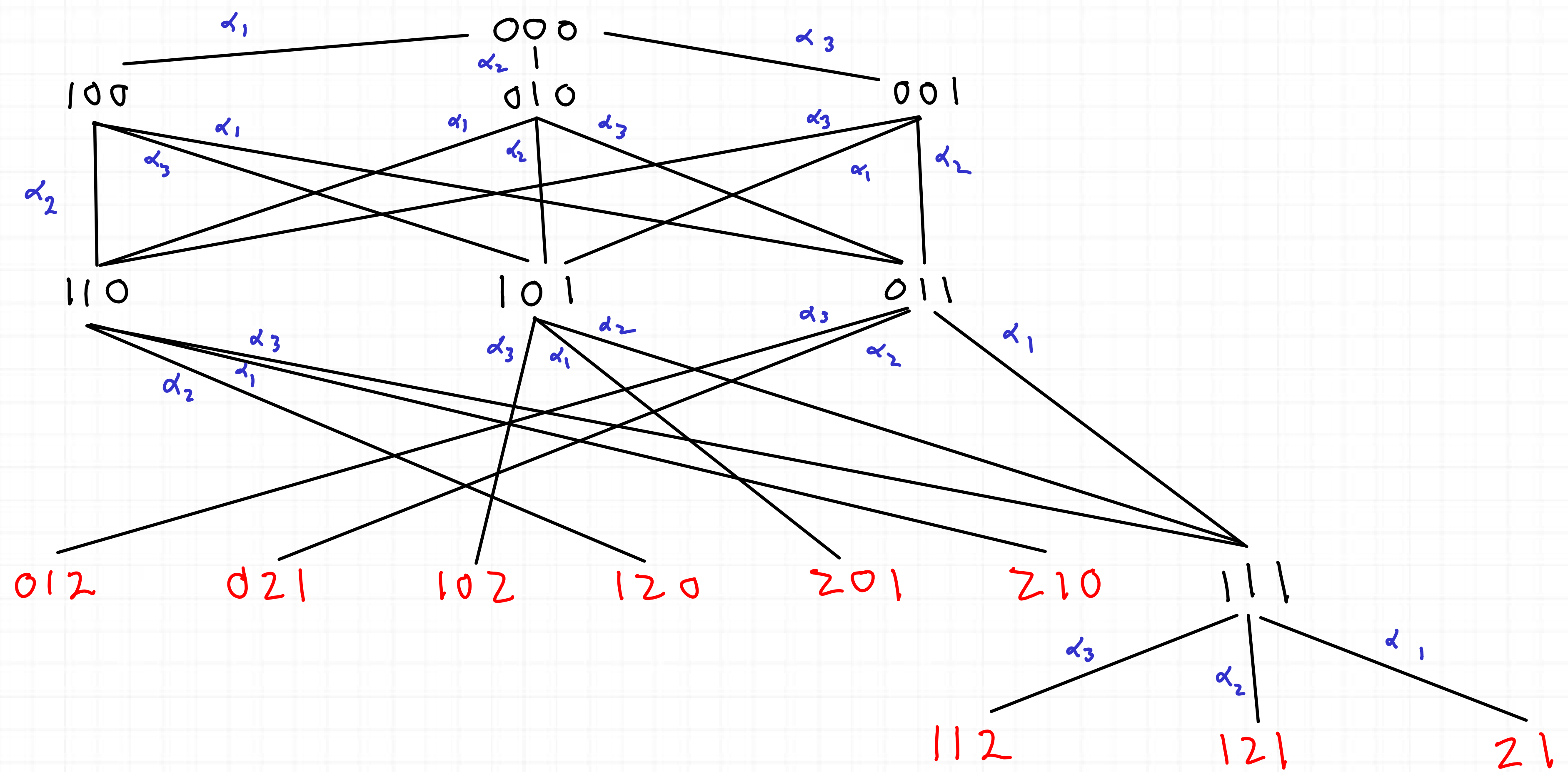
$\zeta_\tau(0000) =$

$$\frac{1}{4}\Big[\underbrace{\left(\tfrac{1}{4}\right)^6\cdot 6\cdot 4!\cdot 60}_{0123} + \underbrace{\left(\tfrac{1}{4}\right)^7\cdot 7\cdot \tfrac{4!}{2}\cdot 272}_{1123} + \underbrace{\left(\tfrac{1}{4}\right)^7\cdot 7\cdot \tfrac{4!}{2}\cdot 256}_{0223} + \underbrace{\left(\tfrac{1}{4}\right)^8\cdot 8\cdot \tfrac{4!}{2}\cdot 1072}_{1223} + \underbrace{\left(\tfrac{1}{4}\right)^9 9\cdot \tfrac{4!}{3!}\cdot 4288}_{2223}\Big]$$

$= \dfrac{7143}{4096} \approx 1.74$ while $\zeta(K_4) = \dfrac{111}{64} \approx 1.73$.

Threshold density $\zeta_\tau(000)$ for K_3

(with arbitrary probability density $\alpha : V_{K_3} \to [0,1]$)



$$1$$

$$\alpha_1 \qquad \alpha_2 \qquad \alpha_3$$

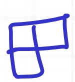
$$2\alpha_1\alpha_2 + \alpha_3^2 \qquad 2\alpha_1\alpha_3 + \alpha_2^2 \qquad \alpha_1^2 + 2\alpha_2\alpha_3$$

$$\alpha_3(2\alpha_1\alpha_2+\alpha_3^2) + \alpha_2(2\alpha_1\alpha_3+\alpha_2^2) + \alpha_1(\alpha_1^2+2\alpha_2\alpha_3)$$

$$(\alpha_2+\alpha_3)(\alpha_1^2+2\alpha_2\alpha_3) + (\alpha_1+\alpha_2)(2\alpha_1\alpha_2+\alpha_3^2) + (\alpha_1+\alpha_3)(2\alpha_1\alpha_3+\alpha_2^2)$$

$$\|$$

$$3\left(\sum \alpha_i^2\alpha_j\right)$$

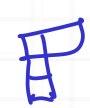
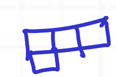
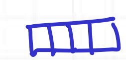
3 

$$+$$

$$(\alpha_1+\alpha_2+\alpha_3)\left[\alpha_3(2\alpha_1\alpha_2+\alpha_3^2)+\alpha_2(2\alpha_1\alpha_3+\alpha_2^2)+\alpha_1(\alpha_1^2+2\alpha_2\alpha_3)\right]$$

$$\|$$

$$6\left(\sum \alpha_i^2\alpha_j\alpha_k\right) + \sum \alpha_i^3\alpha_j + \sum \alpha_i^4$$

6  +  + 

A few new things from this semester

* For a sandpile graph G with n vertices,

$$\det(L+J) = n \cdot \sum_v \tau_v$$

where τ_v is the number of directed spanning trees rooted at v.

* $\zeta(K_n) \sim \frac{n-1}{2} + \frac{\sqrt{2\pi n}}{4} - \frac{7}{6} + O\left(\frac{1}{\sqrt{n}}\right).$

* The symmetrized reduced Laplacian is an M-matrix (probably).

* Alternate proof of Gessel's theorem.