Math 374

Threshold density: $\zeta_\tau(D_0) := \mathbb{E}_{D_0} \dfrac{\deg D_\tau}{\# V}$

stationary density: $\zeta_{st} = \dfrac{1}{\# S(G)} \sum_{c \in S(G)} \dfrac{|c|}{\# V}$ where $|c| = \deg(c) + \deg(s)$.

Thm. (Levine, 2014) $\zeta_\tau(D_0) \to \zeta_{st}$ as $\deg D_0 \to -\infty$.

We have seen that the theorem follows from

$$\lim_{\deg D_0 \to -\infty} \mathbb{P}_{D_0}\big((v_\tau, c_\tau, m_\tau) = (v, c, m) \big) \overset{\star}{=} \frac{\alpha(v)}{\# S(G)} \mathbb{1}\{\deg(s) \le m < \beta_v(c) + \deg(s)\}$$

Our goal today is to prove $\star$.

Recall the Markov renewal theorem:

irreducible Markov chain: $M = (\Omega, P, (X_t))$

edges: $\mathcal{E} := \{(x,y) : P(x,y) > 0\}$

length function: $\ell : \mathcal{E} \to \mathbb{N} = \mathbb{Z}_{\geq 0}$.

aperiodic: Assume $\gcd(\ell(C) : C \text{ a cycle}) = 1$.

stationary distribution: π

cumulative length at time t: $\lambda_t = \sum_{i=1}^{t} \ell(X_{i-1}, X_i)$

nth finishing time: $\tau_n = \min\{t : \lambda_t \geq n\}$.

Markov renewal theorem

Let $x_0, x, y \in \Omega$ be any states, and let $m \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \mathbb{P}_{x_0}\left((X_{\tau_n-1}, X_{\tau_n}, \lambda_{\tau_n} - n) = (x,y,m)\right) = \begin{cases} \frac{1}{Z}\pi(x) P(x,y) & \text{if } 0 \leq m \leq \ell(x,y) - 1 \\ 0 & \text{otherwise} \end{cases}$$

where Z is the normalizing constant $\sum_{(\tilde{x},\tilde{y}) \in \Omega} \pi(\tilde{x}) P(\tilde{x},\tilde{y}) \ell(\tilde{x},\tilde{y})$.

Proof of Levine's theorem

We apply the Markov renewal theorem to the following chain:

$$\Omega = V \times S(G) \quad , \qquad X_t = (v_t, c_t)$$

$$P((v',c'),(v,c)) = \begin{cases} \alpha(v) & \text{if } (c'+v)^\circ = c \\ 0 & \text{otherwise.} \end{cases}$$

If $((v',c'),(v,c)) \in \mathcal{E}$, i.e., if $P((v',c'),(v,c)) > 0$, then define

$$\ell((v',c'),(v,c)) = \beta_v(c) = \deg(c') - \deg(c) + 1.$$

Since $(c+s)^\circ = c$, we have $((v,c),(v,c)) \in \mathcal{E}$, and $\ell((v,c),(v,c)) = 1$.

Therefore, $\gcd\{\lambda(C) : C \text{ cycle}\} = 1$, as required.

Exercise This chain is irreducible, and its stationary distribution is

$$\pi(v,<) = \frac{\alpha(v)}{\# S(G)}.$$

Consider the s-recurrent decomposition for the closed chain:

$$D_t = c_t + (m_t + \deg s)s - L\sigma_t.$$

The threshold time is $\tau = \min\{t : D_t \text{ alive}\} = \min\{t : m_t \geq 0\}$.

Recall that if $D_t = (D_{t-1} + v)^\circ$, then

$$m_t = m_{t-1} + \beta_v(c_t) = m_{t-1} + \ell((v_{t-1}, <_{t-1}), (v_t, <_t))$$

Hence,

$$m_t = m_0 + \sum_{i=1}^{t} \ell((v_{i-1}, c_{i-1}), (v_i, <_i)) = m_0 + \lambda_t.$$

So D_t is alive if $m_0 + \lambda_t \geq 0$. Therefore, the threshold time is $\tau = \min\{t : \lambda_t \geq -m_0\}$. By the Markov renewal theorem:

$$\lim_{m_0 \to -\infty} \mathbb{P}_{D_0}\left((v_{\tau-1}, c_{\tau-1}), (v_\tau, c_\tau), m_\tau \right) = \left((v', c'), (v, c), m \right) \Big)$$

$$= \lim_{m_0 \to -\infty} \mathbb{P}_{D_0}\Big((v_{\tau-1}, c_{\tau-1}), (v_\tau, c_\tau), \lambda_\tau + m_0 \Big) = \Big((v', c'), (v, c), m \Big) \Big)$$

Markov renewal

$$= \frac{1}{Z} \pi(v', c') P((v', c'), (v, c)) \, \mathbb{1}\{ 0 \leq m < \ell((v', c'), (v, c)) \}$$

$$= \frac{1}{Z} \frac{\alpha(v')}{\# S(G)} \alpha(v) \, \mathbb{1}\{ (c' + v)^\circ = c \text{ and } 0 \leq m < \beta_s(v) \}$$

Calculate Z:

$$Z = \sum_{((v', c'), (v, c)) \in \Omega} \pi(v', c') P((v', c'), (v, c)) \, \ell((v', c'), (v, c))$$

$$= \sum_{\substack{((v',c'),(v,c)) \\ \text{s.t. } (v+c')^\circ = c}} \frac{\alpha(v')}{\# S(G)} \alpha(v) \beta_v(c)$$

$\beta_v(c)$

$c' \to (c'+v)^\circ = c$

$$= \sum_{v,v' \in V} \frac{\alpha(v')}{\# S(G)} \alpha(v) \sum_{c' \in S(G)} \beta_v((c'+v)^\circ).$$

However, for each v,

$$\sum_{c' \in S(G)} \beta_v((c'+v)^\circ) = \sum_{c'} \deg(c') - \deg(c' \circledast \eta_v) + 1$$

$$= \sum_{c'} \deg(c') - \sum_{c'} \deg(c' \circledast \eta_v) + \sum_{c'} 1$$

$$= \# S(G).$$

It follows that

$$Z = \sum_{v,v'} \alpha(v')\alpha(v) = 1.$$

Therefore,

$$\lim_{m_0 \to -\infty} \mathbb{P}_{D_0}\Big((v_{\tau-1}, c_{\tau-1}), (v_\tau, c_\tau), m_\tau \big) = \big((v', c'), (v, c), m \big) \Big)$$

$$= \frac{\alpha(v')}{\# S(G)} \alpha(v) \, \mathbb{1}\left\{ (c'+v)^\circ = c \text{ and } 0 \le m < \beta_s(v) \right\}$$

Now sum over (v', c'):

$$\mathbb{P}_{D_0}\big((v_\tau, c_\tau, m_\tau) = (v, c, m) \big)$$

$$\sum_{(v',c')} \mathbb{P}_{D_0}\Big((v_{\tau-1}, c_{\tau-1}), (v_\tau, c_\tau), m_\tau \big) = \big((v', c'), (v, c), m \big) \Big)$$

$$\longrightarrow \sum_{(v',c')} \frac{\alpha(v')}{\# S(G)} \alpha(v) \, \mathbb{1}\left\{ (c'+v)^\circ = c \text{ and } 0 \le m < \beta_s(v) \right\}$$

$$= \sum_{c' \in S(G)} \frac{\alpha(v)}{\# S(G)} \mathbb{1}\left\{ (c'+v)^\circ = c \text{ and } 0 \le m < \beta_s(v) \right\}$$

$\leftarrow$ There is only one non-zero term in this sum: $c' = c \circledast \eta_{-v}$

$$= \frac{\alpha(V)}{\# S(G)} \mathbb{1}\{0 \le m < \beta_s(v)\}.$$

(Note that $m_0 \to -\infty$ gives the same limit as $\deg D_0 \to -\infty$.)