Math 374

Last time: $\mathbb{P}_{D_0}(\deg D_\tau = n) \to \frac{1}{\# S(G)} \# \{ c \in S(G) : |c| = n \}$

S_4

	n	$\mathbb{P}$
$(012 + 3s) \times 6$	6	6/16
$(022 + 3s) \times 3$, $(112 + 3s) \times 3$	7	6/16
$(122 + 3s) \times 3$	8	3/16
$(222 + 3s) \times 1$	9	1/16

$t_{K_4}(y) = 6 + 6y + 3y^2 + y^3$

Therefore,

$$\mathbb{E}_{D_0}(\deg D_\tau) = \sum_n n \, \mathbb{P}_{D_0}(\deg D_\tau = n) = \frac{1}{\# S(G)} \sum_n n \; \# \{ c \in S(G) : |c| = n \}$$

$$= \frac{1}{\# S(G)} \sum_{c \in S(G)} |c|$$

It follows that the threshold density, $\zeta_\tau(D_0) := \frac{1}{\# V} \mathbb{E}_{D_0}(D_\tau)$, approaches the stationary density, $\zeta(G) := \frac{1}{\# S(G)} \frac{1}{\# V} \sum_{c \in S(G)} |c|$ as $\deg D_0 \to -\infty$.

Corollary: $\lim_{\deg D_0 \to -\infty} \zeta_\tau(D_0) = \zeta(G)$.

Our goal now is to prove Levine's theorem,

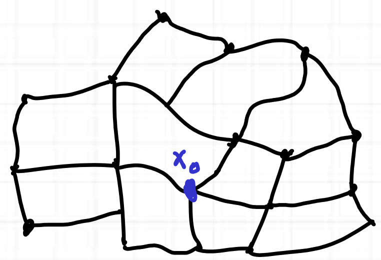
Amt. of same to sink in

$c' \to (c'+v)^{\circ} = c$

$$\mathbb{P}_{D_0}\left((v_\tau, c_\tau, m_\tau) = (v, c, m)\right) \longrightarrow \begin{cases} \dfrac{\alpha(v)}{\# S(G)} & \text{if} \quad \deg(s) \le m < \deg(s) + \overbrace{\beta_v(c)} \\[2ex] 0 & \text{otherwise.} \end{cases}$$

The theorem is a consequence of a "Markov renewal" theorem we now describe.

Consider the street map of town pictured below:



Each road connecting intersection x to intersection y has a nonnegative integer length: $\ell(x,y) \in \mathbb{N}$. Now imagine the following Markov marathon race of length $n \in \mathbb{Z}$. Runners start at intersection x_0. Throughout the race, when a runner reaches intersection x, she chooses a road (x,y) at random (based on a probability transition matrix), then runs to intersection y along that road. She keeps track of her total length: $\lambda_{t+1} = \lambda_t + \ell(x,y)$. She finishes the race when her total length is at least n.

Question: What is the probability that her finishing road is (x,y)?

What is the excess distance she has run at that time: $\lambda - n$?

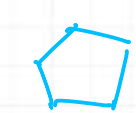
total distance $- n$

General set-up

$M = (\Omega, P, (X_t))$ an irreducible Markov chain

edges: $\mathcal{E} := \{(x,y) : P(x,y) > 0\}$

length function: $\ell : \mathcal{E} \to \mathbb{N} = \mathbb{Z}_{\geq 0}$.



If C is a cycle of edges in $\mathcal{E}$, define $\ell(C) = \sum_{(x,y) \in C} \ell(x,y)$

Assume $\gcd\{\ell(c) : c \text{ a cycle}\} = 1$. This means for all sufficiently large L, there is a closed path in $\mathcal{L}$ of length L.

Let π be the stationary distribution for M (which exists and is unique since M is irreducible).

Cumulative length at time t: $\lambda_t = \sum_{i=1}^{t} \ell(X_{i-1}, X_i)$.

For each $n \in \mathbb{N}$, define the n^{th} finish time:

$$\tau_n = \min\{t : \lambda_t \geq n\}.$$

Markov renewal theorem

Let $x_0, x, y \in \Omega$ be any states, and let $m \in \mathbb{N}$. Then

$$\lim_{n \to \infty} \mathbb{P}_{x_0}\left((X_{\tau_n - 1}, X_{\tau_n}, \lambda_{\tau_n} - n) = (x, y, m)\right) = \begin{cases} \frac{1}{Z} \pi(x) P(x,y) & \text{if } 0 \leq m \leq \ell(x,y) - 1 \\ 0 & \text{otherwise} \end{cases}$$

where Z is the normalizing constant $\sum_{(\tilde{x},\tilde{y})\in\Omega} \pi(\tilde{x})\, P(\tilde{x},\tilde{y})\, \ell(\tilde{x},\tilde{y})$.