Math 374

* Compute threshold density $\zeta_\tau(\vec{0})$ on K_4.

Stable effective divisors of each degree, up to symmetry

degree   divisor   possible threshold state

0	0000			
1	0001			
2	0011	0002		
3	0111	0012		
4	1111	0112	0022	
5	1112	0122		
6	1122	0222		0123
7	1222			1123 0223
8	2222			1223
9				2223

For $D \in \mathrm{Div}(G)$, let $\lambda(D)$ be the number of possible paths from $\vec{0}$ to D in the closed chain (D_t) on K_4. Then

$$\zeta_\tau(\vec{0}) = \frac{1}{4}\left(\binom{4}{1,1,1,1}\lambda(0123)\left(\tfrac{1}{4}\right)^6\cdot 6 + \binom{4}{1,1,2}\lambda(1123)\left(\tfrac{1}{4}\right)^7\cdot 7 + \binom{4}{1,1,2}\lambda(0223)\left(\tfrac{1}{4}\right)^7\cdot 7\right.$$

$$\left. + \binom{4}{1,1,2}\lambda(1223)\left(\tfrac{1}{4}\right)^8\cdot 8 + \binom{4}{1,3}\lambda(0223)\left(\tfrac{1}{4}\right)^9\cdot 9\right)$$

where $\binom{4}{i_1 \cdots i_k} = \dfrac{4!}{i_1! \cdots i_k!}$.

number of permutations of 0223

number of paths to 0023

probability of any path of length 9

Try to compute $\lambda(D)$ recursively. For example, which divisors can precede 0112 in the chain? They must be among permutations of 0111 and 0012.

Note: $(0012 + v_4)^\circ = (0013)^3 = 1102$. So $(2001 + v_1)^\circ = (3001)^\circ = 0112$

So exactly $\{0012, 0102, 0111, 2001\}$ precede 0112. Therefore,

$$\lambda(0112) = \lambda(0012) + \lambda(0102) + \lambda(0111) + \lambda(2001) = \lambda(0111) + 3\lambda(0012).$$

Back to Levine's theorem:

Thm. (Levine, 2014) Consider our Markov chains:

$$D_t = c_t + (m_t + \deg_G(s))s - L\sigma_t .$$

Let $\tau = \tau(D_0)$ be the threshold time, and let v_τ be the threshold vertex.

Then,

$$\lim_{\deg D_0 \to -\infty} \mathbb{P}_{D_0}\big((v_\tau, c_\tau, m_\tau) = (v, c, m)\big) = \frac{\alpha(v)}{\# S(G)} \mathbb{1}\{\deg(s) \le m < \beta_v(c) + \deg(s)\}$$

$$= \begin{cases} \dfrac{\alpha(v)}{\# S(G)} & \text{if } \deg(s) \le m < \beta_v(c) + \deg(s) \\ 0 & \text{otherwise.} \end{cases}$$

Last time:

1. $$\lim_{\deg D_0 \to -\infty} \mathbb{P}_{D_0}(c_\tau = c) = \frac{1}{\# S(G)} \sum_{v \in V} \alpha(v)\beta_v(c)$$

② $\mathbb{P}_{D_0}(v_\tau = s,\ c_\tau = c) \longrightarrow \dfrac{\alpha(s)}{\# S(G)}$

$\Rightarrow \mathbb{P}_{D_0}(v_\tau = s) \longrightarrow \alpha(s)$

$\Rightarrow \mathbb{P}_{D_0}(v_\tau = v) \longrightarrow \alpha(v)$

Continue:

④ $\mathbb{P}_{D_0}(\deg D_\tau = n) \longrightarrow \dfrac{1}{\# S(G)} \# \{c \in S(G) : |c| = n\}$

Pf/ We have

$$\mathbb{P}_{D_0}(\deg D_\tau = n) \overset{\star}{=} \sum_v \mathbb{P}_{D_0}(v_\tau = v,\ \deg D_\tau = n).$$

To compute $\mathbb{P}_{D_0}(v_\tau = v = \deg D_\tau = n)$, consider the v-recurrent decomposition

$$D_\tau = c_\tau^{(v)} + (m_\tau^{(v)} + \deg_G(s))\, v - L\sigma_\tau^{(v)}.$$

Since D_τ is alive, $m_\tau^{(v)} \geq 0$

Since $D_{\tau-1}$ is not alive and $m_\tau^{(v)} = \underbrace{m_{\tau-1}^{(v)}}_{<\deg(s)} + \beta_v(c_\tau^{(v)}) = m_{\tau-1}^{(v)} + 1$,

From earlier calculation of decomposition of $a_u D_t$.

we have $m_\tau^{(v)} = -1$.

Hence, $\deg(D_\tau) = \deg c_\tau^{(v)} + \deg_G(v) = |c_\tau^{(v)}|$. Therefore,

$$\mathbb{P}_{D_0}(v_\tau = v, \deg D_\tau = n) = \mathbb{P}_{D_0}(v_\tau = v, |c_\tau| = n)$$

$$= \sum_{\substack{c \in S(G,v) \\ \text{s.t. } |c| = n}} \mathbb{P}_{D_0}(v_\tau = v, c_\tau = c)$$

$$\xrightarrow{②} \sum_{\substack{c \in S(G,v) \\ \text{s.t. } |c| = n}} \frac{\alpha(v)}{\# S(G)}$$

$$= \frac{\alpha(v)}{\# S(G)} \, \# \{ c \in \overbrace{S(G,v)}^{\text{recurrents w.r.t. sink } v} : |c| = n \}.$$

If G is undirected, then $\# \{ c \in S(G,v) : |c| = n \}$ is independent of v.

If G is Eulerian, the number is still independent of v due to a result of Perrot and Pham. Thus, from ☆, we get

$$\mathbb{P}_{D_0}(\deg D_\tau = n) = \sum_v \mathbb{P}_{D_0}(v_\tau = v, \ \deg D_\tau = n)$$

$$\to \sum_v \frac{\alpha(v)}{\# S(G)} \# \{c \in S(G) : |c| = n\}$$

$$= \frac{1}{\# S(G)} \# \{c \in S(G) : |c| = n\}$$

Example $G = K_4$. $\deg(s) = 3$

increasing recurrents	$\lvert c \rvert$	$\lim_{\deg(D_0) \to -\infty} \mathbb{P}_{D_0}(\deg D_\tau = n)$
012	6	$\frac{1}{16} \cdot 1 \cdot 3! = \frac{6}{16}$
022	7	$\frac{1}{16} \cdot 2 \cdot 3 = \frac{6}{16}$
112	7	
122	8	$\frac{1}{16} \cdot 1 \cdot 3 = \frac{3}{16}$
222	9	$\frac{1}{16} \cdot \underbrace{1}_{\# \lvert c \rvert = n} \cdot 1 = \frac{1}{16}$