Math 374

$G = (V, E, s)$ Eulerian sandpile graph, $\alpha : V \to [0,1]$ prob. dist., $\alpha(v) > 0$ $\forall v$.

$D \in \mathrm{Div}(G)$: recurrent decomposition $D = c + ms - L\sigma$, $c \in \mathcal{S}(G)$, $\sigma(s) = 0$.

Markov chains:

(D_t) closed: $a_v D = \begin{cases} D + v & \text{if } D+v \text{ alive} \\ (D+v)^\circ & \text{otherwise} \end{cases}$

(c_t) open $\tilde{a}_v c = (c+v)^\circ$ w/ $(c+s)^\circ := c$

Last time $a_v D = (c+v)^\circ + (m + \deg_G(s) + \beta)s - L(\sigma + \mathrm{odo}(c+v) + \mu - \mu(s))$

$\beta = \beta_v(\underbrace{(c+v)^\circ}_{\tilde{a}_v c})$

firing script for $c+v \to (c+v)^\circ$

firing script for $D+v \to (D+v)^\circ$ if $D+v$ stabilizable, otherwise $\mu = 0$.

burst size for $c \in \mathcal{S}(G)$, $v \in V$: $\beta_v(c) := \deg(\mathrm{rec}(-v) \circledast c) - \deg(c) + 1$

$\beta_v(c)$ = amount of sand going into the sink in the stabilization

$$(\mathrm{rec}(-v) \circledast c) + v \longrightarrow c.$$

Example: See table for K_4 from yesterday. We found that

$$\sum_{c \in S(K_4)} \sum_{v \in V} \beta_v(c) = 64.$$

Conjecture: In general, for each $v \in V$, $\sum_{c \in S(G)} \beta_v(c) = \# S(G)$

Thm. (Levine, 2014) Consider our Markov chains:

$$D_t = c_t + (m_t + \deg_G(s))s - L\sigma_t.$$

Let $\tau = \tau(D_0)$ be the threshold time, and let v_τ be the threshold vertex.

Then,

$$\lim_{\deg D_0 \to -\infty} \mathbb{P}_{D_0}\big((v_\tau, c_\tau, m_\tau) = (v, c, m)\big) = \frac{\alpha(v)}{\# S(G)} \mathbb{1}\{0 \le m < \beta_v(c)\}$$

$$= \begin{cases} \dfrac{\alpha(v)}{\# S(G)} & \text{if } 0 \le m < \beta_v(c) \\ 0 & \text{otherwise.} \end{cases}$$

First remarks.

* Why would the probability be zero if $m \ge \beta_v(c)$?

Consider the state, D', of the closed chain one step before reaching $D_\tau = c + (m + \deg_G(s))s - L\sigma$. Say $D' = c' + (m' + \deg_G(s))s - L\sigma$. Then

$$a_v D' = (c'+v)^\circ + (m' + \deg_G(s) + \beta_v((c'+v)^\circ))s - L(*) = D.$$

Hence, $c = (c'+v)^\circ$ and $m = m' + \beta_v(c)$. So if $m \ge \beta_v(c)$, then $m' \ge 0$, and D' is already alive.

Corollaries.

1. Sum over v and m to get

$$\lim_{\deg D_0 \to -\infty} \mathbb{P}_{D_0}(c_\tau = c) = \frac{1}{\# S(G)} \sum_{v \in V} \alpha(v)\, \beta_v(c)$$

Example $G = K_4$,

Take $\alpha(v) = \frac{1}{4}$ $\forall v$.

Then $\mathbb{P}_{D_0}(c_\tau = 012) \to \frac{1}{16} \cdot 7$

$\mathbb{P}_{D_0}(c_\tau = 222) \to \frac{1}{16}$

etc.

recurrents up to symmetry	β_{v_0}	β_{v_1}	β_{v_2}	β_{v_3}	permutations
012	1	1	2	3	$\times 6$
022	1	1	0	0	$\times 3$
112	1	0	0	3	$\times 3$
122	1	0	0	0	$\times 3$
222	1	0	0	0	$\times 1$

See computer simulation.

2. $$\mathbb{P}_{D_0}(v_\tau = s, c_\tau = c) = \sum_m \mathbb{P}_{D_0}((v_\tau, c_\tau, m_\tau) = (s, c, m))$$

$$\to \frac{\alpha(s)}{\# S(G)} \beta_s(c) = \frac{\alpha(s)}{\# S(G)}.$$

BTW

$\beta_s(c) = 1$; hence, there is only one term in the sum: $m = \deg(s)$.

Summing over $c \in S(G)$, we get

$$\mathbb{P}_{D_0}(v_\tau = s) \to \alpha(s)$$

But the evolution of (D_t) is independent of the choice of sink.

We could re eat the above argument for any choice of sink.

Hence,

$$\mathbb{P}_{D_0}(v_\tau = v) \longrightarrow \alpha(v)$$

for all $v \in V$.