Math 374

$G = (V, E, s)$ Eulerian sandpile graph

$\mathrm{Config}(G) := \mathbb{Z}\tilde{V} \subseteq \mathbb{Z}V = \mathrm{Div}(G).$

We are interested in two Markov chains:

Closed Chain: $\Omega = \mathrm{Div}(G)$, (D_t), $\alpha : V \to [0,1]$, $\alpha(v) > 0 \ \forall v$

$$a_v D = \begin{cases} (D+v)^\circ & \text{if } D+v \text{ stabilizable} \\ D+v & \text{otherwise } (D+v \text{ is alive}) \end{cases}$$

$$D_0 \xrightarrow{v_1} a_{v_1} D = D_1 \xrightarrow{v_2} a_{v_2} D_1 = D_2 \longrightarrow \cdots$$

Open chain $\Omega = S(G)$, (c_t), $\alpha : V \to [0,1]$, $\alpha(v) > 0 \ \forall v$

$$\tilde{a}_v c = (c+v)^\circ \quad \text{where } (c+s)^\circ := c.$$

Next goal: Compare these chains.

Given $D \in \mathrm{Div}(G)$, define $D|_{s=0} = c$ where $D = c + ks$ with $c \in \mathrm{Config}(G)$. Let $\mathrm{rec}(D)$ be the recurrent sandpile equivalent to $D|_{s=0}$ to get a homomorphism

$$\mathrm{rec}: \mathrm{Div}(G) \longrightarrow S(G)$$

In particular, $\mathrm{rec}(s)$ is the identity of $S(G)$.

Recurrent decomposition of a divisor

We have seen that if $D \in \mathrm{Div}(G)$, then $\exists!\, c \in S(G)$ and $!\, m \in \mathbb{Z}$ s.t.

$$D \sim c + ks = c + (m + \deg(s))s$$

and D is unstable iff $m \geq 0$.

This means $D = c + (m + \deg_G(s))\, s - L\sigma$ for some firing script σ. By adding an appropriate multiple of $\vec{1}$, the generator of $\ker L$, to σ, we may assume $\sigma(s) = 0$.

In sum, $\exists$ unique $c \in S(G)$, $m \in \mathbb{Z}$, and $\sigma \in Script(G)$ with $\sigma(s) = 0$ such that

$$D = c + (m + \deg_G(s))\, s - L\sigma$$

} recurrent decomposition of D (w.r.t. s)

and D is stabilizable iff $m < 0$.

Now think about how the recurrent decomposition behaves with respect to our Markov chains. We will need to keep track of how much sand goes into the sink at each step

$$\begin{array}{ccccccccc} c_0 & \longrightarrow & c_1 & \longrightarrow & c_2 & \longrightarrow & c_3 & \longrightarrow & \cdots \\ & & \| & & \| & & & & \\ & & \tilde{a}_{v_1} c_0 & & \tilde{a}_{v_2} c_1 & & & & \end{array}$$

Define the burst size of $c \in S(G)$ at v to be the number of grains of sand that fall into the sink in the stabilization of $R_s(-v) + c$:

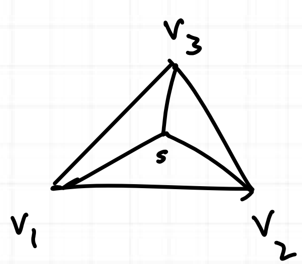
$$\beta_v(c) = \deg(\underbrace{\mathrm{rec}(-v) \circledast c}_{=\tilde{a}_v^{-1} c}) - \deg(c) + 1.$$

"av" for "avalanche"

$= \tilde{a}_v^{-1} c$ since $\mathrm{rec}(-v) \circledast c + v \sim c + v$ and $(\mathrm{rec}(-v) \circledast c + v)^{\circ}$ is recurrent.

So β_v can be thought of as the amount of sand going into the sink in one step of our chain: $\tilde{a}_v^{-1} c \xrightarrow{v} c$

Example K_4



recurrents up to symmetry	β_{v_0}	β_{v_1}	β_{v_2}	β_{v_3}
0 1 2	1	1	2	3
0 2 2	1	1	0	0
1 1 2	1	0	0	3
1 2 2	1	0	0	0
2 2 2	1	0	0	0

$c_{max} = 222 =$ identity

(why?)

$-100 = -100 + 222 \mod \tilde{\mathcal{L}}$

$= 122$

$\Rightarrow R_s(-v_1) = 122$

Question. What patterns can we find for the burst sizes? Any nice combinatorics hidden here?

Thm. If $D = c + (m + \deg_G(s))s - L\sigma$ is the recurrent decomposition of D, then

$$a_v D = (c+v)^\circ + (m + \deg_G(s) + \beta)s + L(\sigma + \mathrm{odo}(c+v) + \mu - \mu(s))$$

where

- $\beta = \beta_v((c+v)^\circ)$.
- $\mathrm{odo}(c+v)$ is the firing script for the stabilization of $c+v$.

  We take $[\mathrm{odo}(c+v)](s) = 0$.

  $$(c+v)^\circ = c + v - L(\mathrm{odo}(c+v)).$$

  If $v = s$, then $(c+v)^\circ := c$, and $\mathrm{odo}(c+v) = 0$. So

  $$(c+s)^\circ = c - L(\mathrm{odd}(c+v)).$$

- μ is the firing script for the stabilization of $D+v$, taking $\mu = 0$ if $D+v$ is alive.

Pf/ We have

$$
\begin{aligned}
a_v D &= D + v - L\mu \\
&= c + (m + \deg_G(s))s - L\sigma + v - L\mu \\
&= c + v + (m + \deg_G(s))s - L(\sigma + \mu) \\
&= c + v + L\,\mathrm{odo}(c+v) + (m + \deg_G(s))s - L(\sigma + \mu - \mathrm{odo}(c+v))
\end{aligned}
$$

In the stabilization $c + v \to (c+v)^\circ$, the amount of sand going into the sink is $\beta_v((c+v)^\circ)$. Note $\beta_s((c+s)^\circ) = 1$.

Therefore, $a_v D = (c+v)^\circ + (m + \deg_G(s) + \beta)s - L(\sigma + \mu - \mathrm{odo}(c+v))$. $\square$