Math 374

$G = (V, E, s)$ undirected sandpile graph, $g = \# E - \# V + 1$

$h = (h_0, \ldots, h_g)$, $h_i = \#$ superstables of degree i.

Merino. $t_G(y) := T(G; 1, y) = \sum_{i=0}^{g} h_{g-i}\, y^i$.

Cor. $y^g\, t_G\left(\frac{1}{y}\right) = \sum_{i=0}^{g} h_i y^i$.

$\therefore \sum_{i=0}^{g} i h_i y^{i-1} = \left(y^g\, t_G\left(\frac{1}{y}\right)\right)'$

$= g y^{g-1}\, t_G\left(\frac{1}{y}\right) + y^g \cdot \frac{-1}{y^2}\, t_G'\left(\frac{1}{y}\right)$.

Setting $y = 1$: $\sum_{c \text{ superstable}} \deg(c) = g\, t_G(1) - t_G'(1)$ [Note: $t_G(1) = \# S(G)$.]

The stationary density is $\zeta(G) = \frac{1}{\# S(G)} \frac{1}{\# V} \sum_{c \in S(G)} |c|$, where $|c| = \deg(c) + \deg_G(s)$.

Bijection: c superstable $\Longleftrightarrow$ $c_{max} - c$ recurrent.

Therefore, $$\sum_{c \in S(G)}' |c| = \sum_{c \text{ superstable}} |c_{max} - c| = \sum_{c \text{ s.s.}}' \left[\deg(c_{max} - c) + \deg_G(s)\right]$$

$$= \# S(G) \left[\deg(c_{max}) + \deg_G(s)\right] - \sum_{c \text{ s.s.}} \deg(c)$$

$$= t_G(1)\left[\deg(c_{max}) + \deg_G(s)\right] - \left(g\, t_G(1) - t_G'(1)\right)$$

$$= t_G(1)\left[\# E + g\right] - \left(g\, t_G(1) - t_G'(1)\right)$$

$$= (\# E)\, t_G(1) + t_G'(1)$$

$\deg c_{max}$

$= \sum_{v \in \tilde{V}} (\deg(v) - 1)$

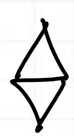
$= \sum_{v \in V} (\deg(v) - 1) - \deg(s) + 1$

$= 2(\# E) - \# V - \deg(s) + 1$

$= \# E + g - \deg(s)$

Result: $$\sum_{c \in S(G)} |c| = (\# E)\, t_G(1) + t_G'(1).$$ Divide by $(\# E)(\# S(G)) = (\# E)\, t_G(1)$:

Stationary density: $$\zeta(G) = \frac{1}{\# V}\left[\# E + \frac{t_G'(1)}{t_G(1)}\right] = \frac{1}{\# V}\left[\# E + \ln(t_G(y))'\Big|_{y=1}\right]$$

Example:  See previous notes.

By homework, $t_G'(1)$ is the number of spanning unicycles of G.
A unicycle of G is a connected subgraph of G with a unique cycle.
A spanning unicycle of G is a unicycle containing all of the vertices of G, equivalently, it's a subgraph of G with $\#V$ vertices and $\#V$ edges (thus, obtained by adding an edge to a spanning tree).

Therefore, the we can express the stationary density as

Stationary density: $$\zeta(G) = \frac{1}{\#V}\left(\#E + \frac{\#\text{ spanning unicycles}}{\#\text{ spanning trees}}\right)$$

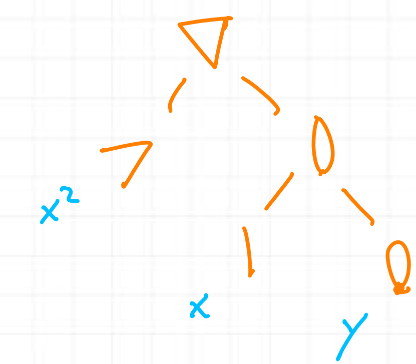
Complete Graph Let $T_n(x,y)$ be the Tutte polynomial of the complete graph, K_n.

Gessel/Pak: $$T_n(x,y) = \sum_{k=0}^{n-2} \binom{n-2}{k}(x+y+y^2+\cdots+y^k)\, T_{k+1}(1,y)\, T_{n-k-1}(x,y)$$

where $T_1(x,y) := 1$ and $(x+y+\cdots+y^0) := x$.

So $T_2(x,y) = x\, T_1(1,y)\, T_1(x,y) = x$ and

$$T_3(x,y) = x\, T_1(1,y)\, T_2(x,y) + (x+y)\, T_2(1,y)\, T_1(x,y)$$
$$= x^2 + x + y.$$



Thoughts:

* The Tutte polynomial of $\overset{k+1}{\bullet\!\!-\!\!\bullet}$ ($k+1$ edges) is $x+y+\cdots+y^k$
* If $f = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ and $g = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!}$, then the coefficient of $\frac{x^n}{n!}$ in fg is $\sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r}$.

Stationary density of K_n

See Sage for a plot of $\zeta(K_n)$ as a function of n. The points seem to lie on the line

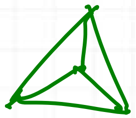
$$y = 0.605186x - 0.601045$$

Tutte (+ Gessel/Sagan)

$$\sum_{n\geq 1} T_n(x,y)\frac{z^n}{n!} = \frac{1}{x-1}\left[\left[\sum_{n\geq 0} y^{\binom{n}{2}}(y-1)^{-n}\frac{z^n}{n!}\right]^{(x-1)(y-1)} - 1\right]$$

Can we calculate (estimate) $\frac{\partial}{\partial y} T_n(1,1)$ from this?

Example K_4 superstables = parking function $- \vec{1}$



$c_{max} = 222$

increasing superstables	increasing recurrents	# permutations		$\lvert c \rvert = \deg(c) + 3$
012	012	6	×	6
002	022	3	×	7
011	112	3	×	7
001	122	3	×	8
000	$222 = c_{max}$	1	×	9

deg(s)

$$\frac{1}{\# S(K_4)} \frac{1}{\# V} \sum_{c \in S(K_4)} |c| = \underbrace{\frac{1}{4^2}}_{n^{n-2}} \frac{1}{4} (36 + 21 + 21 + 24 + 9) = \underbrace{\frac{1}{4^3}}_{n^{n-1}} (111) = \frac{111}{64}$$

$$\zeta(K_n) = \frac{1}{\# V}\left[\# E + \frac{t'_{K_n}(1)}{t_{K_n}(1)}\right] = \frac{1}{n}\left[\binom{n}{2} + \frac{t'_{K_n}(1)}{n^{n-2}}\right]$$

$$= \frac{n-1}{2} + \frac{t'_{K_n}(1)}{n^{n-1}}.$$

It turns out that

$$t'_{K_n}(1) = n^{n-\frac{1}{2}}\left(\sqrt{2\pi}\cdot\frac{1}{4} - \frac{7}{6}\frac{1}{n^{1/2}} + \sqrt{2\pi}\,\frac{1}{48}\,\frac{1}{n} + \frac{131}{270}\,\frac{1}{n^{3/2}} + \sqrt{2\pi}\,\frac{1}{1152}\,\frac{1}{n^2} - \frac{4}{2835}\,\frac{1}{n^{5/2}} + O\left(\frac{1}{n^3}\right)\right).$$

So $$\zeta(K_n) = \frac{n-1}{2} + \frac{\sqrt{2\pi n}}{4} - \frac{7}{6} + \frac{\sqrt{2\pi}}{4}\,\frac{1}{n^{1/2}} + \frac{131}{270}\,\frac{1}{n} + \cdots + O\left(\frac{1}{n^{5/2}}\right),$$

and therefore,

$$\zeta(K_n) \sim \frac{n-1}{2} + \frac{\sqrt{2\pi n}}{4} - \frac{7}{6} + O\left(\frac{1}{\sqrt{n}}\right).$$