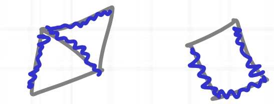
Math 374 Merino's theorem

$G = (V, E)$ undirected multigraph, not necessarily connected.



$\deg K = 2g^* - 2.$

- $\mathrm{rank}(G) :=$ # edges in a maximal forest of G = n − # components of G

- Tutte polynomial: $$T(x,y) = T(G;x,y) := \sum_{A \subseteq E} (x-1)^{\mathrm{rank}(E) - \mathrm{rank}(A)} (y-1)^{\#A - \mathrm{rank}(A)}$$

- $e \in E$ is a bridge if its remove increases the number of components

Thm.

(1) $T(G; x,y) = 1$ if $E = \emptyset$.

(2) $T(G; x,y) = y\,T(G \setminus e; x,y)$ if e is a loop.

(3) $T(G; x,y) = x\,T(G/e; x,y)$ if e is a bridge

(4) $T(G; x,y) = T(G \setminus e; x,y) + T(G/e; x,y)$ if e is neither a loop nor a bridge

Pf/ Exercise. $\square$

Bonus

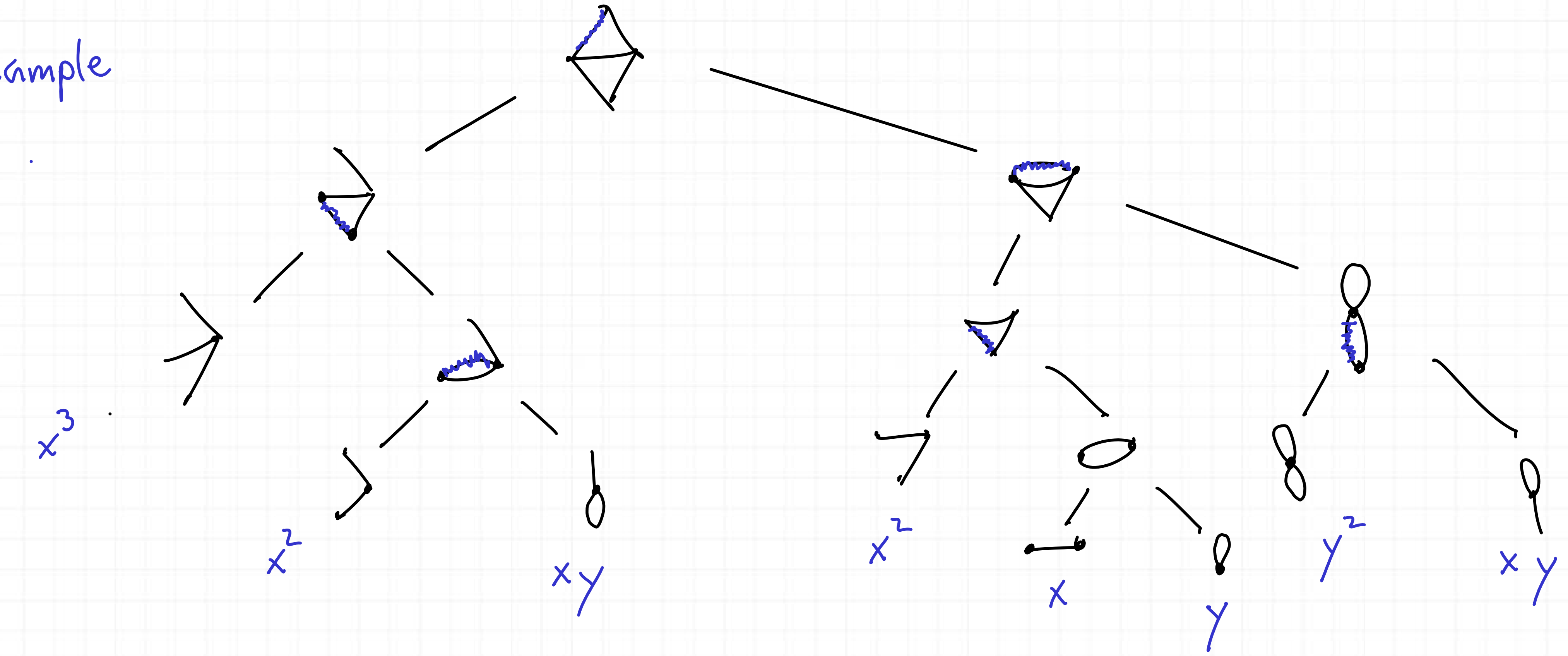
$T(1,1)$ = # spanning forests

$T(2,1)$ = # forests

$T(1,2)$ = # spanning subgraphs

$T(2,0)$ = # acyclic orientations

Example



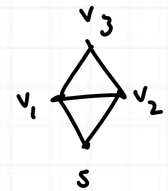
$$T(x,y) = x + 2x^2 + x^3 + (1+2x)y + y^2.$$

Independent of decomposition!

Thm. (Merino, 1997) $G=(V,E,s)$ undirected sandpile graph.

$g = \#E - \#V + 1$, $\quad h = (h_0, \dots, h_g)$ where $h_i = \#$ superstables of degree i

Then,
$$T(1,y) = \sum_{i=0}^{g} h_{g-i}\, y^i.$$

Example $G =$ 

superstables:	degree
200, 101, 011, 020	2
100, 010, 001	1
000	0

$g = 2$

$h = (h_0, h_1, h_2)$
$= (1, 3, 4)$

$$T(1,y) = 4 + 3y + y^2$$

Pf/ By induction on the number of edges. Define $h_0 = 1$ if G has only a single vertex. Therefore, if G has no edges, the result holds: $g = 0$ and $T(x,y) = h_0 = 1$.

Suppose $e \in E$ and $s \in e$. Three cases:

I. Suppose e is a loop. Let $G' = G \setminus e$. Then the genus of G' is $g' = g - 1$ and the superstables on G and G' are the same, so their h-vectors are the same: $h = h'$.

We have $T(G;x,y) = y\,T(G\setminus e;x,y) \Rightarrow T(G;1,y) = y\,T(G';1,y)$.

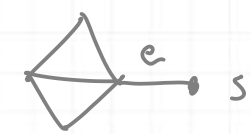
By induction

$$T(G;1,y) = y\,T(G';1,y) = y\sum_{i=0}^{g'} h'_{g'-i}\, y^i = \sum_{i=0}^{g-1} h_{g-1-i}\, y^{i+1}$$

$h_g = 0$ ← maximal superstables on an undirected multigraph with loops is $g - \#$ loops.

$$= \sum_{i=1}^{g} h_{g-i}\, y^i = \sum_{i=0}^{g} h_{g-i}\, y^i.$$

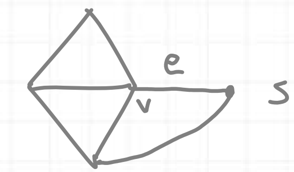
II Suppose e is a bridge, and let



$G' = G/e$ with sink s. Then $g' = g$ and $h' = h$. We have $T(G;x,y) = x\,T(G/e;x,y)$. Therefore, by induction:

$$T(G;1,y) = T(G';1,y) = \sum_{i=0}^{g'} h'_{g-i}\, y^i = \sum_{i=0}^{g} h_{g-i}\, y^i.$$

III. Suppose $e = \{s, v\}$ is neither loop nor bridge.



Partition the superstables:

$$\mathcal{A} = \{ c \text{ superstable} : c(v) = 0 \}$$

$$\mathcal{B} = \{ c \text{ superstable} : c(v) > 0 \}.$$

Claims:

① $\mathcal{A}$ is the set of superstables of G/e. ← Where we think of $c \in \text{Config}(G/e)$ as $c \in \text{Config}(G)$ by letting $c(v) = 0$.

Pf/ Let $c \in \text{Config}(G)$ with $c(v) = 0$, and let $W \subseteq \tilde{V} = V \setminus \{s\}$ s.t. it is legal to fire W from c. Since $c(v) = 0$ and v is connected to $s \notin W$, we have $v \notin W$. So c is superstable if $\nexists\ \emptyset \neq W \subseteq V \setminus \{s, v\}$.

② $\mathcal{B}$ is in bijection with the superstables of $G \setminus e$ via:

$c \in \mathcal{B} \longrightarrow c^- \in \mathrm{Config}(G \setminus e)$

$$c^-(u) = \begin{cases} c(u) & \text{for } u \neq v \\ c(v)-1 & \text{for } u = v. \end{cases}$$

Pf/ Clear. $\square$

Therefore, $h_i(G) = h_i(G/e) + h_{i-1}(G \setminus e)$. So by induction,

$$\begin{aligned} T(G;1,y) &= T(G/e;1,y) + T(G\setminus e;1,y) \\ &= \sum_{i=0}^{g} h_{g-i}(G/e)y^i + \sum_{i=0}^{g-1} h_{g-1-i}\, y^i \\ &= \sum_{i=0}^{g} h_{g-i}(G)\,y^i. \quad \square \end{aligned}$$

Recall, the stationary density of G is $\zeta(G) = \frac{1}{\# S(G)} \sum_{c \in S(G)} \frac{|c|}{\# V}$

where $|c| = \deg(c) + \deg(s)$.

We have c recurrent iff $c_{max} - c$ superstable, and

$$\deg(c_{max} - c) = \sum_{v \in \tilde{V}} (\deg(v) - 1) - \deg(c)$$

$$= \sum_{v \in V}' (\deg(v) - 1) - \deg(c) - (\deg(s) - 1)$$

$$= 2(\# E) - \# V - \deg(c) - \deg(s) + 1$$

$$= \# E + g - |c|.$$

Then, $h_i = \#$ superstables of degree i

$$= \# \{ c \in S(G) : |c| = g - i + (\# E) \}.$$

$$\Rightarrow h_{g-i} = \# \{ c \in S(G) : |c| = i + (\# E) \}.$$

Then $y^{\# E} T(G; 1, y) = \sum_{i=0}^{g} h_{g-i} \, y^{i + (\# E)}$

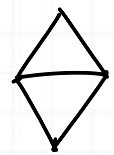
$$\Rightarrow \frac{\partial}{\partial y} \left(y^{\# E} \, T(G; 1, y) \right) \Big|_{y=1} = \left[\sum_{i=0}^{g} (i + \#(E)) \, h_{g-i} \, y^{i + (\# E) - 1} \right]_{y=1}$$

$$\Rightarrow \#(E) \, T(G; 1, 1) + \frac{\partial}{\partial y} T(G; 1, y) \Big|_{y=1} = \sum_{i=0}^{g} (i + \#(E)) \left[\# \{ c \in S(G) : |c| = i + \#(E) \} \right]$$

$$\Rightarrow \#(E)\,\#S(G) + \frac{\partial}{\partial y} T(G;1,y)\Big|_{y=1} = (\#S(G))(\#V)\,\zeta(G)$$

$$\Rightarrow \zeta(G) = \frac{1}{\#S(G)}\cdot\frac{1}{\#V}\,\frac{\partial}{\partial y}T(G;1,y)\Big|_{y=1} + \frac{\#E}{\#V}.$$

$$= \left[\left.\frac{\frac{\partial}{\partial y}T(G;x,y)}{T(G;x,y)}\right|_{x=y=1} + \#E\right]\frac{1}{\#V}.$$

Example.  , $T = 4+3y+y^2$, $T_y = 3+2y$, $T_y|_{y=1} = 5$

$$\zeta(G) = \left[\frac{5}{8} + 5\right]\frac{1}{4} = \frac{45}{32}.$$