Math 374

$G = (V, E, s)$ an Eulerian sandpile graph.

$\alpha : V \longrightarrow [0,1]$ probability distribution with $\alpha(v) > 0 \quad \forall v$.

Markov process: state space $= \mathrm{Div}(G)$

- Pick v at random according to α.
- $D \longmapsto a_v D := \begin{cases} (D+v)^\circ & \text{if } D+v \text{ stabilizable} \\ D+v & \text{otherwise} \end{cases}$ adding a grain of sand

Formally, the transition function is

$$P(D, D') = \begin{cases} \alpha(v) & \text{if } D' = D + v \text{ and } D' \text{ is alive} \\ \sum\limits_{\substack{v \text{ s.t.} \\ (D+v)^\circ = D'}} \alpha(v) & \text{if } D' \text{ is stable.} \end{cases}$$

Starting at $X_0 = D_0$, let $X_{t+1} = a_{v_t} X_t$ for $t \geq 0$ where $v_0, v_1, v_2, \ldots$

are independent random draws from α.

There are no essential or recurrent states.

Define the random time $\tau = \tau(D_0) := \min\{t \geq 0 : X_t \text{ alive}\}$,

Then $\mathbb{P}(\tau = t) = \sum \prod_{i=0}^{t} \alpha(v_i)$ where the sum is over all strings $v_1 \cdots v_t$ where D_t is the first alive divisor in the corresponding sequence

$$D_0 \xrightarrow{v_1} \underset{\substack{\| \\ \alpha_{v_1} D_0}}{D_1} \xrightarrow{v_2} \underset{\substack{\| \\ \alpha_{v_2} D_1}}{D_2} \xrightarrow{v_3} \cdots \xrightarrow{v_t} D_t .$$

We may then consider the random divisor D_τ.

We have $\mathbb{P}(D_\tau = D)$ is $\sum \prod_{i=1}^{k} \alpha(v_i)$

where the sum is over all strings $v_1 \cdots v_k$ of all lengths $k \geq 0$ such that $D_0 \xrightarrow{v_1} \cdots \xrightarrow{v_k} D_k$ with $D_k = D$ and D is the first alive divisor in the sequence.

Define the random vertex v_τ which refers to the vertex where sand is added resulting for the first time in an alive divisor. We have $\mathbb{P}(v_\tau = v) = \sum \prod \alpha(v_i)$ where the sum is over all words $v_0 \cdots v_k$ of all lengths $k \geq 0$ such that $D_0 \xrightarrow{v_0} \cdots \xrightarrow{v_k} D_k$ with D_k the first alive divisor in the sequence and $v_k = v$.

Questions:

- What is the threshold density: $\zeta_\tau(D_0) := \mathbb{E}_{D_0} \dfrac{\deg D_\tau}{\# V}$, i.e. the expected density of sand (per vertex) of D_τ (starting at D_0).

- $\mathbb{P}(\deg D_\tau = k)$ for each k?
- $\mathbb{P}(v_\tau = v)$ for each v?

Def. The **stationary density** of G is

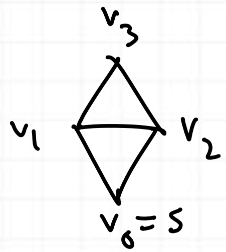
$$\zeta_{st} = \frac{1}{\# S(G)} \sum_{c \in S(G)} \frac{|c|}{\# V}$$

where $|c| = \deg(c) + \deg(s)$.

Note that for each $D \in \mathrm{Div}(G)$, we have $D \sim c + ks$ for a unique $c \in S(G)$ and $k \in \mathbb{Z}$. Then D is alive iff $k \geq \deg(s)$.

Thm. (Levine, 2014) $\zeta_\tau(D_0) \to \zeta_{st}$ as $\deg D_0 \to -\infty$.

Example



superstables $\begin{smallmatrix}&0&\\2&&0\end{smallmatrix}$ $\begin{smallmatrix}&1&\\1&&0\end{smallmatrix}$ $\begin{smallmatrix}&1&\\0&&1\end{smallmatrix}$ $\begin{smallmatrix}&0&\\0&&2\end{smallmatrix}$ $\quad c_{max} = \begin{smallmatrix}&1&\\2&&2\end{smallmatrix}$

$\begin{smallmatrix}&0&\\1&&0\end{smallmatrix}$ $\begin{smallmatrix}&1&\\0&&0\end{smallmatrix}$ $\begin{smallmatrix}&0&\\0&&1\end{smallmatrix}$

$\begin{smallmatrix}&0&\\0&&0\end{smallmatrix}$

$\big\downarrow \; c_{max} - c$

recurrents

				$\lvert c \rvert$
$\begin{smallmatrix}&1&\\0&&2\end{smallmatrix}$	$\begin{smallmatrix}&0&\\1&&2\end{smallmatrix}$	$\begin{smallmatrix}&0&\\2&&1\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&0\end{smallmatrix}$ ★	5
$\begin{smallmatrix}&1&\\1&&2\end{smallmatrix}$	$\begin{smallmatrix}&0&\\2&&2\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&1\end{smallmatrix}$		6
$\begin{smallmatrix}&1&\\2&&2\end{smallmatrix}$				7

Stationary density:

$$\zeta_{st} = \frac{1}{8} \cdot \frac{1}{4} \left(4 \cdot 5 + 3 \cdot 6 + 1 \cdot 7 \right)$$

$$= \frac{45}{32}$$

What about the threshold density with $D_0 = \vec{0}$ and, say, $\alpha(v) = \frac{1}{4}$ $\forall v$?

One possible path:

$$\begin{smallmatrix}&0&\\0&&0\\&0&\end{smallmatrix} \xrightarrow{\frac14} \begin{smallmatrix}&0&\\1&&0\\&0&\end{smallmatrix} \xrightarrow{\frac14} \begin{smallmatrix}&0&\\2&&0\\&0&\end{smallmatrix} \xrightarrow{\frac14} \begin{smallmatrix}&0&\\3&&0\\&0&\end{smallmatrix} \sim \begin{smallmatrix}&1&\\0&&1\\&1&\end{smallmatrix} \xrightarrow{\frac14} \begin{smallmatrix}&1&\\1&&1\\&1&\end{smallmatrix} \xrightarrow{\frac14} \begin{smallmatrix}&1&\\2&&1\\&1&\end{smallmatrix} \xrightarrow{\frac14} \begin{smallmatrix}&1&\\3&&1\\&1&\end{smallmatrix}$$

degree = 6

$\leftarrow$ alive

$\sim \begin{smallmatrix}&2&\\0&&2\\&2&\end{smallmatrix} \sim \begin{smallmatrix}&0&\\1&&3\\&2&\end{smallmatrix} \sim$ ★ $\begin{smallmatrix}&1&\\2&&0\\&3&\end{smallmatrix}$

This contributes $\dfrac{\left(\frac{1}{4}\right)^6 \cdot 6}{4}$ $\leftarrow \deg(D_\tau)$ $\leftarrow \#V$ to ζ_τ and $\left(\frac{1}{4}\right)^6 \cdot 6$ to $\mathbb{E}(\tau)$.