Math 374

$G = (V, E, s)$, Eulerian sandpile graph, $D \in \mathrm{Div}(G)$.

- $v \in V$ is stable in D if $D(v) < \deg(v)$.
- D is stable if v is stable in D for all v, otherwise, D is unstable.
- D is alive if every diviser in its linear equivalence class, $[D]$ is unstable.

  Note: Given any D, there is some $D' \sim D$ with D' unstable: just pick any $v \in V$ and have that vertex borrow enough times.
- D is minimal alive if D is alive and $D' \lneq D \Rightarrow D'$ not alive.

This is equivalent to saying D is alive and $\forall v \in V$, $D - v$ is not alive.

- D is maximal unwinnable if D is unwinnable and $D + v$ is winnable $\forall v \in V$. Equivalently, D is unwinnable and $D \lneq D' \Rightarrow D'$ winnable.

Def. $D_{max} = K - \vec{1} = \sum_{v \in V} (\deg(v) - 1) v$

Prop. D is (maximal) unwinnable iff $D_{max} - D$ is (minimal) alive.

Pf/$(\Rightarrow)$ Suppose D is unwinnable and let $F \sim D_{max} - D$. Then

$(D_{max} - F) \sim D \Rightarrow \exists v \in V$ s.t. $(D_{max} - F)(v) < 0 \Rightarrow D_{max}(v) < F(v)$

$\Rightarrow F$ unstable.

Suppose D is maximal unwinnable, let $D^* := D_{max} - D$, and let $v \in V$.
We need to show $D^* - v$ is not alive. By maximality, $D + v \sim E \geq 0$.
Hence, $D^* - v = D_{max} - (D + v) \sim D_{max} - E$, and $D_{max} - E$ is stable. So D^* is not alive.

$(\Leftarrow)$ Similar.

Cor. If D is alive, then D is minimal alive iff $\deg D = \# E$.

Pf/ D is minimal alive iff $D^* := D_{max} - D$ is maximal unwinnable.

iff D^* unwinnable and $\deg D^* = g-1 = \#E - \#V$. Then $\deg D^* = g-1$

iff $\deg D = \deg(D_{max} - D^*) = \deg D_{max} - g + 1 = 2(\#E) - \#V - (\#E - \#V) = \#E.$ $\square$

Cor. D alive iff $D \sim r + ks$ with r recurrent and $k > \deg(s)$.

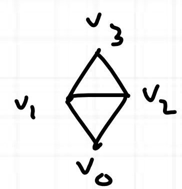
Pf/ D alive iff $D^* := D_{max} - D$ unwinnable iff $D^* \sim c - \ell q$ with

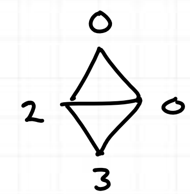
c superstable and $\ell > 0$ $\iff$ $D \sim (c_{max} + (\deg(s) - 1)s) - (c + \ell q)$

$$= \underbrace{(c_{max} - c)}_{r} + (\underbrace{\deg(s) + \ell - 1}_{k})s. \quad \square$$

Cor. D alive $\Rightarrow$ D winnable, i.e., $|D| \neq \emptyset$.

Let $\mathrm{Fire}(D)$ be the direct graph with vertex set $|D|$ and edges (D', D'') if $\exists$ unstable v in D' s.t. $D' \xrightarrow{v} D''$.

Example $G =$ 

, $D =$ 

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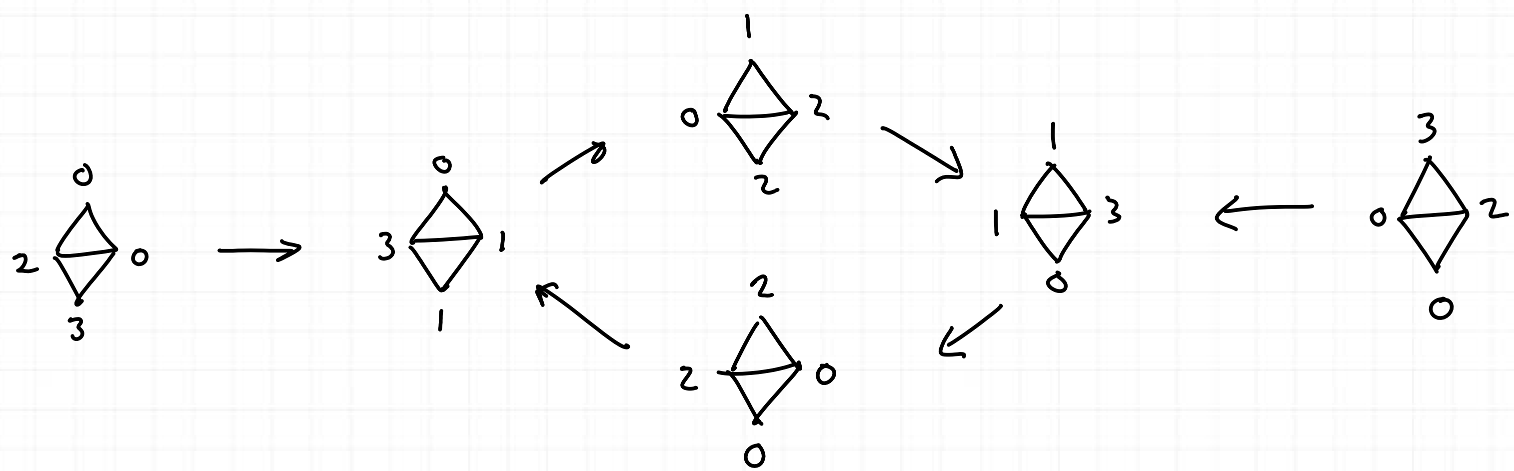

Prop. D alive $\Rightarrow$ the undirected graph underlying $\mathrm{Fire}(D)$ is connected.

Pf/ Say $D = c + ks$ for some $c \in \mathrm{Config}(G)$ and $k \in \mathbb{Z}$. Let $b := \tilde{L}\vec{1} \in \mathrm{Config}(G)$, the configuration obtained by firing s.

Through vertex firings, we have $D \to c^\circ + k's$ with c° stable. Since D is alive, s is unstable in $c^\circ + k's$. Hence,

$$c^\circ + k's \xrightarrow{s} c^\circ + b + k''s.$$

Stabilizing again:

$$c^\circ + b + k''s \to (c^\circ + b)^\circ + k'''s$$

with s unstable in $(c^\circ + b)^\circ + k'''s$. Repeating ℓ times, we get

$$D \to (((c^\circ + b)^\circ + b)^\circ + \cdots)^\circ + \tilde{k}s$$

where $\tilde{k} > \deg(s)$. By the abelian property,

$$(((c^\circ + b)^\circ + b)^\circ + \cdots)^\circ = (c + \ell b)^\circ$$

By taking ℓ large enough there will be vertex firings s.t.

$\ell b \longrightarrow c_{max} + a$ with $a \geq 0$. Hence,

$(c + \ell b)^\circ = (c_{max} + c + a)^\circ$ is recurrent.

We have shown that for any alive divisor, D, $\exists$ vertex firings s.t. $D \to r + ks$ with r recurrent and $k > \deg(s)$.
If $D' \in |D|$, then D' is alive, so $D' \sim r' + k's$. However, since $D' \sim D$, we have $r' = r$ and $k's$. Thus, ever $D' \in |D|$ has a path to $r + ks$ in $\mathrm{Fire}(D)$. $\square$