Math 374

$G = (V, E)$ directed multigraph, $V = \{1, \ldots, n\}$, Laplacian L

- $\tau_i :=$ # spanning trees of G rooted at i.

Suppose at least one τ_i is nonzero, and define

$\tau := (\tau_1, \ldots, \tau_n)$ and $\tilde{\tau} = \tau / \gcd(\tau)$.

Prop. $\ker L = \mathbb{Z}\tau$.

Pf/ The rank of L is $n-1$. (The row vectors sum to zero, so $\operatorname{rk} L < n$, but there is a spanning tree directed into some vertex, so by the matrix-tree theorem some minor of size $n-1$ is nonzero. Thus, $\operatorname{rk} L \geq n-1$.) Therefore, it suffices to show $(\tau_1, \ldots, \tau_n) \in \ker L$.

Expand $\det L$ along its i^{th} row:

$$0 = \det L = \sum_{j=1}^{n} L_{ij} (-1)^{j+1} \det L^{(ij)}$$

$$= \sum_{j=1}^{n} L_{ij} \det L^{(ij)}$$ (since the row vectors of L sum to $\vec{0}$, by HW)

$$= \sum_{j=1}^{n} L_{ij} \tau_j$$ (matrix-tree). □

Let J be the $n \times n$ matrix of all 1s: $J_{ij} = 1 \quad \forall i,j$.

If M is an $n \times n$ matrix, define the adjugate of M to be the $n \times n$ matrix $\operatorname{adj}(M)$ with

$$\operatorname{adj}(M)_{ij} = (-1)^{i+j} \det M^{(ji)}$$

$M^{(ji)}$: remove the j^{th} row and i^{th} column from M.

Then for $n \times n$ matrices M and N, we have:

(i) $\operatorname{adj}(MN) = \operatorname{adj}(N)\operatorname{adj}(M)$

(ii) $\operatorname{adj}(M)\,M = M\operatorname{adj}(M) = \det M \cdot I_n$.

Prop. $\det(L+J) = n\left(\sum_{i=1}^{n} \tau_i\right)$.

Pf/ Since $JL = O$ and $J^2 = nJ$, we have

$$(nI_n - J)(L+J) = nL.$$

Therefore, $\operatorname{adj}(L+J)\operatorname{adj}(nI_n - J) = \operatorname{adj}(nL) = n^{n-1}\operatorname{adj} L$.

Since $nI_n - J$ is the Laplacian matrix for K_n, we have $\operatorname{adj}(nI_n - J) = n^{n-2} J$ by the matrix-tree theorem.

It follows that

$$\operatorname{adj}(L+J)\,J = n \operatorname{adj} L. \qquad (\star)$$

By matrix tree, $\operatorname{adj} L = \begin{bmatrix} - \tau_1 - \\ \vdots \\ - \tau_n \end{bmatrix}$ = $n \times n$ matrix w/ i^{th} row $\tau_i, \dots, \tau_i$.

From $(\star)$,

$$\det(L+J)\,J = (L+J)\operatorname{adj}(L+J)\,J = n(L+J)\begin{bmatrix} - \tau_1 - \\ \vdots \\ - \tau_n - \end{bmatrix}$$

$$= nL\begin{bmatrix} - \tau_1 - \\ \vdots \\ - \tau_n - \end{bmatrix} + nJ\begin{bmatrix} - \tau_1 - \\ \vdots \\ - \tau_n - \end{bmatrix}$$

$$= 0 + n\left(\Sigma \tau_i\right) J,$$

and the result follows. □