Math 374

$G = (V, E, s)$ a sandpile graph

For $v, w \in \tilde{V}$ and $c \in S(G)$, let $n(v,w;c)$ be the number of times w topples in the stabilization of $c + v$. Then

$$(\tilde{L}^{-1})_{wv} = \frac{1}{\# S(G)} \sum_{c \in S(G)} n(v,w;c) = \text{the average number of times } w \text{ topples when a grain of sand is added to } v.$$

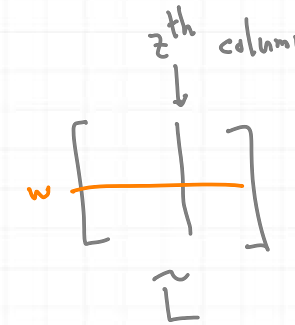
Pf/ Define the $\#\tilde{V} \times \#\tilde{V}$ matrix N by

$$N_{vw} = \frac{1}{S(G)} \sum_{c \in S(G)} n(v,w,c).$$

We need to show that $N\tilde{L} = I$, the identity matrix.

We have $(c+v)^\circ(w) = c(w) + \underbrace{\delta(v,w)}_{} - \sum_{z\in\tilde{V}} n(v,z;c)\,\tilde{L}_{wz}$

For the grain of sand added to c at v to begin with.



Averaging over $S(G)$:

$$\frac{1}{\#S(G)}\underbrace{\sum_{c\in S(G)}(c+v)^\circ(w)} = \underbrace{\frac{1}{\#S(G)}\sum_{c\in S(G)}c(w)} + \delta_v(w) - \frac{1}{\#S(G)}\sum_{c\in S(G)}\sum_{z\in\tilde{V}} n(v,z;c)\,\tilde{L}_{wz}$$

$$= \frac{1}{\#S(G)}\sum_{c\in S(G)}(c * \underset{\text{recurrent equivalent to } v}{\eta_v})(w)$$

cancel

$$= \frac{1}{\#S(G)}\sum_{c\in S(G)}c(w)$$

$$\Longrightarrow \delta_v(w) = \sum_{z\in\tilde{V}}\left(\frac{1}{\#S(G)}\sum_{c\in S(G)} n(v,z;c)\right)\tilde{L}_{wz} = \sum_{z\in\tilde{V}} N_{vz}\,\tilde{L}^t_{zw} = (N\tilde{L}^t)_{vw}. \qquad \square$$

Example: See Sage.

Question: Variance?

Thm. If $M = (\Omega, P, (X_t))$ is an irreducible Markov chain with stationary distributions μ and π, then $\mu = \pi$.

Pf/ Since $\pi P = \pi$, we have $\pi(P-I) = 0$. So the rank of $P-I$ is at most $\#\Omega - 1$. If we can show the rank is exactly $\#\Omega - 1$, then $\mu(P-I) = 0 \Rightarrow \mu$ and ı are scalar multiples of each other. But since they are both probability distributions, this would imply $\mu = \pi$, as desired.

To show $\mathrm{rk}(P-I) = 0$, it suffices to show $\dim(\ker(P-I)) = 1$.

First note that $\vec{1} \in \ker(P-I)$ since $P\vec{1} = \vec{1}$ (the entries in any row sum to 1). So we wish to show that the only elements of the kernel are constant vectors.

Suppose $f : \Omega \to \mathbb{R}$ is in the kernel of $P-I$. Let f achieve its maximum at $x \in \Omega$, and let y be an arbitrary state.

Since M is irreducible, $\exists$ path $x = x_0, x_1, x_2, \dots, x_n = y$ with $P(x_i, x_{i+1}) > 0$ for $0 \le i \le n-1$. Now,

$$x \to \begin{bmatrix} & P & \end{bmatrix} \begin{bmatrix} f \end{bmatrix}$$

$$(Pf)(x) = \sum_{z \in \Omega} P(x,z) f(z) \le f(x) \sum_{z \in \Omega} P(x,z) = f(x)$$

with equality iff $f(z) = f(x)$ $\forall z \in \Omega$ s.t. $P(x,z) > 0$. Since, in fact, $Pf = f$ and $P(x_i, x_{i+1}) > 0$ for $0 \le i \le n-1$, we have

$$f(x) = f(x_0) = f(x_1) = \cdots = f(x_n) = f(y).$$

Since y was arbitrary, we see that f is constant. $\square$

On a random walk on a group (H, β) s.t. $\operatorname{supp}(\beta)$ is a set of generators for H, the chain is irreducible, so the uniform distribution is the unique stationary state.

Q: Mixing time?