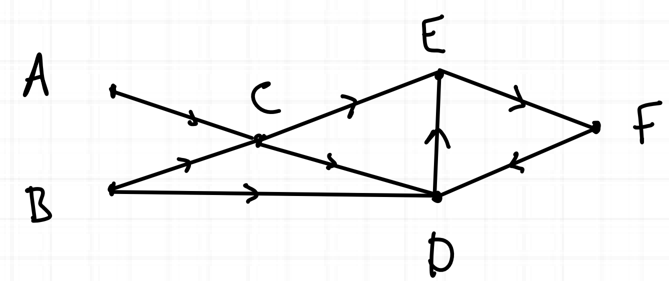
Math 374

$M = (\Omega, P, (X_t))$ Markov chain

$x \to y$ and $y \to x$

Recall vocabulary from last time: accessible ($x \to y$), communication ($x \sim y$), irreducible (! communicating class), essential (x is essential if $\forall y,\ x \to y \Rightarrow y \to x$)

Example from text:

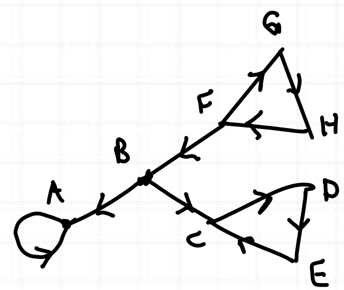


communicating classes: $\{A\}, \{B\}, \{C\}, \{D, E, F\}$
essential: D, E, F.

Def. $x \in \Omega$ is recurrent if when started at x, the chain returns to x with probability 1. Otherwise, x is transient.

Thm. 12.11 For a finite Markov chain (which is the only kind we are considering), $x \in \Omega$ is recurrent iff it's essential. There exists an essential state. If $y \in \Omega$ is inessential, there exists an essential state accessible from it.

Example



communicating classes: $\{A\}, \{B\}, \{C,D,E\}, \{F,G,H\}$

essential: A, C, D, E

Prop. For the sandpile Markov chain on G, there is a unique communicating class, $[c_{max}] = S(G)$, consisting of exactly the essential states.

Pf/ Recall $c \in S(G)$, the sandpile group, exactly when $c = (c_{max} + b)^\circ$ for some $b \geq 0$. Now c_{max} is accessible from every stable configuration, but the elements accessible from c_{max} are exactly the elements of $S(G)$. $\square$

Def. A probability distribution $\pi : \Omega \to [0,1]$ is a stationary distribution for M if $\pi P = \pi$.

Prop. 12.13. If π is a stationary distribution, then $\pi(x) = 0$ if x is inessential.

Thus, for the sake of discussing stationary distributions, we may restrict the chain to the essential states. For the sandpile Markov chain, there is just one essential class, $[c_{max}] = S(G)$. So restricting to the essential states gives irreducible Markov chain. It is an example of something called a random walk on a group.

Random walks on groups

Let H be a finite group and let $\beta : H \to [0,1]$ be a probability distribution. Define a Markov chain with states $\Omega = H$. Starting at $h \in H$, choose $k \in H$ at random according to β, then move to kh. Thus, the transition matrix is given by $P(h,h') = \beta(h'h^{-1})$.

$$h \xrightarrow{\beta(h'h^{-1})} h'$$

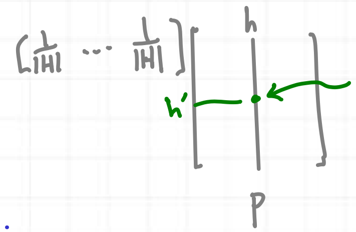
Note that P is a transition matrix, i.e., the elements in any row sum to zero:

since mult. on the right by h^{-1} permutes the elements of H.

$$\sum_{h'\in H} P(h,h') = \sum_{h'\in H} \beta(h'h^{-1}) = \sum_{k\in H} \beta(k) = 1.$$

Thm. H a group, $\beta : H \to [0,1]$ any probability distribution, $\mu : H \to [0,1]$ the uniform distribution: $\mu(h) = \frac{1}{|H|}$ $\forall h \in H$. Then μ is a stationary distribution for the random walk on H with respect to β.

Pf/ $(\mu P)h = \sum_{h' \in H} \frac{1}{|H|} P(h,h') = \frac{1}{|H|} \sum_{h' \in H} \beta(h' h^{-1}) = \frac{1}{|H|} \sum_{k \in H} \beta(k) = \frac{1}{|H|}$.



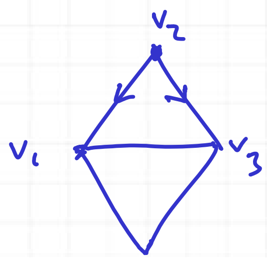
$\square$

the essential/recurrent states/configurations

Sandpile Markov chain (restricted to $S(G)$)

For each $v \in \tilde{V}$, let η_v be recurrent equivalent to v modulo $\tilde{L}$, and let $\eta_s = e$, the recurrent identity. We pick vertices at random according to the distribution $\alpha : V \to [0,1]$. However, it could be that $\eta_v = \eta_{v'}$ even if $v \neq v'$. So let $\beta : S(G) \to [0,1]$ by $\beta(\eta_v) = \sum_{\eta_{v'} = \eta_v} \alpha(\eta_{v'})$ and

$\beta(c) = 0$ if $c \notin \{\eta_v : v \in V\}$. Then the sandpile Markov chain on $S(G)$ w.r.t. α is the random walk on the group $S(G)$ w.r.t. to β.



G

supertables:
$$\begin{matrix} 000 & 100 & 001 \\ 010 & 110 & 011 \end{matrix} \quad \Bigg\| \quad S(G) = \begin{matrix} 111 & 011 & 110 \\ 101 & 001 & 100 \end{matrix}$$

recurrent equivalent to v_2

s, identity $\quad v_3 \quad v_1$

$c_{max} = 111$

$$\tilde{L} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$\alpha : V \to [0,1]$

$[\alpha_1, \alpha_2, \alpha_3, \alpha_s]$

$\beta : S(G) \to [0,1]$

$\beta(100) = \alpha_1$, $\beta(111) = \alpha_2$, $\beta(001) = \alpha_3$

$\beta(101) = \alpha_s$, $\beta(\text{other}) = 0$.