Markov chains

Math 374

* Quickly review sandpile set-up.

* See table in book for sandpile experiment.

* Sandpile Markov chain: $G = (V, E, s)$ a sandpile graph

directed multigraph w/ globally accessible sink $s \in V$

Stab(G): stable configurations on G

$\alpha : V \to [0,1]$ a probability distribution on V (i.e., $\sum_{v \in V} \alpha(v) = 1$).

Assume $\alpha(v) > 0$ $\forall v \in V$, including $v = s$.

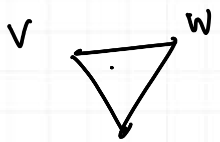
Start with any sandpile c, possibly $c = 0$.

(i) Use α to pick a random vertex v

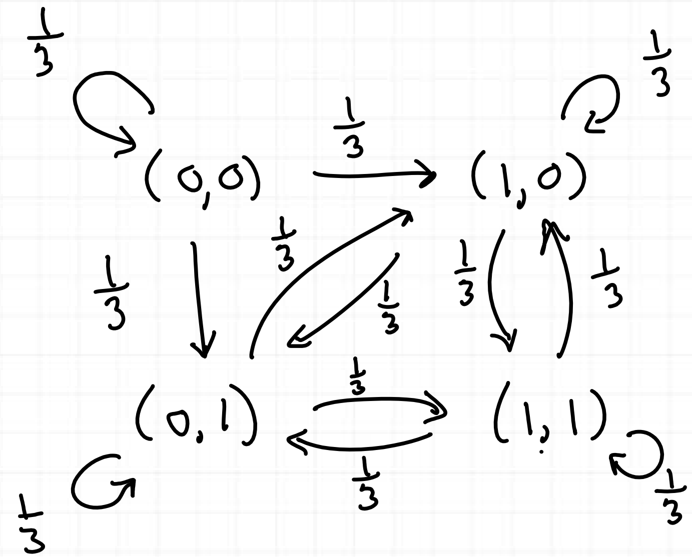
(ii) Replace c be $(c+v)^\circ$, the stabilization of $c+v$.

If $v = s$, define $(c+v)^\circ = c$.

Example



G



$\alpha = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Transition matrix

$$P = \begin{array}{c} \\ 00 \\ 10 \\ 01 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 10 & 01 & 11 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{array}$$

$P(x,y) = P_{xy}$ = probability of transitioning from config. x to config. y.

Starting at configuration $c = \vec{0}$, the probabilities of being at various configurations after one step is

$$\overset{00\ \ 10\ \ 01\ \ 11}{(1,0,0,0)}\,P = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right).$$

After two steps:

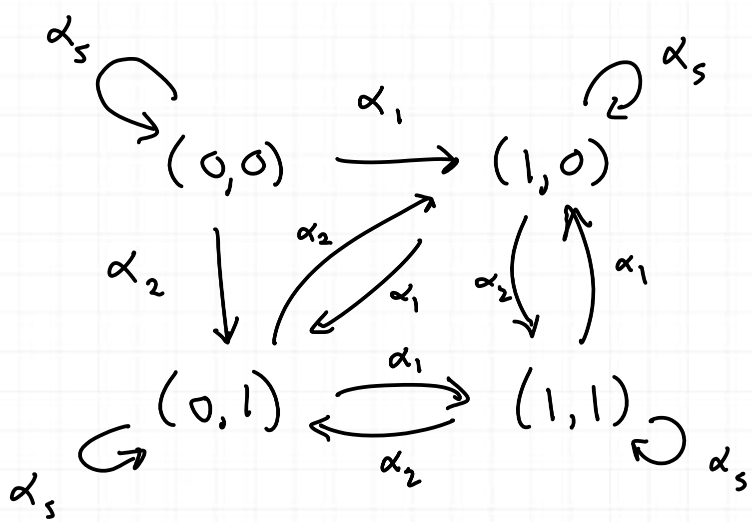
$(1,0,0,0)P^2$

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)P = \left(\frac{1}{9}, \frac{3}{9}, \frac{3}{9}, \frac{2}{9}\right) = \left(\frac{1}{9}, \frac{1}{3}, \frac{1}{3}, \frac{2}{9}\right).$$

After k steps: $(1,0,0,0)\,P^k$.

So the natural question is whether $\lim_{k\to\infty} P^k$ exists.

What if we change α? What if we change the initial configuration?



$$\alpha = (\alpha_s, \alpha_1, \alpha_2)$$

$s \quad v_1 \quad v_2$

Transition matrix

$$P = \begin{array}{c} \\ 00 \\ 10 \\ 01 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 10 & 01 & 11 \end{array} \\ \begin{bmatrix} \alpha_s & \alpha_1 & \alpha_2 & 0 \\ 0 & \alpha_s & \alpha_1 & \alpha_2 \\ 0 & \alpha_2 & \alpha_s & \alpha_1 \\ 0 & \alpha_1 & \alpha_2 & \alpha_s \end{bmatrix} \end{array}$$

☆ Use Sage to check out P^∞ for various values of α.

Def. A finite Markov chain consists of the following:

(i) A finite set of states, Ω.

(ii) A transition matrix $P : \Omega \times \Omega \to [0,1]$ s.t. $\forall x \in \Omega$, the function $P(x,\cdot) : \Omega \to [0,1]$ is a probability distribution: $\sum_{y \in \Omega} P(x,y) = 1$.

(iii) A sequence of random variables $(X_0, X_1, X_2, \dots)$ satisfying $\forall t \geq 0$

$$\mathbb{P}(X_{t+1} = y \mid X_t = x) = P(x,y).$$

Remark:

$$X_t : \mathbb{N}^{\Omega} \to \Omega .$$
$$(s_i)_{i \in \mathbb{N}} \mapsto s_t$$

For each Markov chain $M = (\Omega, P, (X_t))$ there is an associated directed graph $G_M = (V, E)$ with $V = \Omega$ and $E = \{(x,y) : P(x,y) > 0\}$. We think of the edge $(x,y) \in E$ as labeled with $P(x,y)$. The sum of the labels emanating from any vertex x is 1.

If $\pi_t : \Omega \to [0,1]$ be probability distribution giving probabilities for each of the states at step t, then the distribution at step $t+1$ is $\pi_{t+1} = \pi_t P$. It follows that for $t \geq 0$,

$$\pi_t = \pi_0 P^t.$$

Vocabulary $x, y \in \Omega$.

accessible: y is accessible from x if $\exists k \geq 0$ s.t. $P^k(x,y) > 0$. $(x \to y)$

communicating: x and y communicate if x is accessible from y and y is accessible from x. $(y \to x$ and $x \to y)$

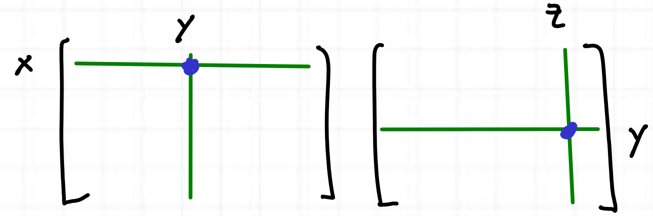
Prop. Communication is an equivalence relation

Pf/ reflexivity: Let $P^0(x,y) = 1 > 0$.

symmetry: By definition.

transitivity: $x \to y$ $\qquad$ $y \to z$

Say $V = \{1, \dots, n\}$.



$= P^{k+k'}$

$P^k(x,y) > 0 \qquad P^{k'}(y,z) > 0$

Let $\ell = y$.

$$P^{k+k'}(x,z) = \left(P^k \cdot P^{k'}\right)(x,z) = \sum_{\ell=1}^{n} P^k(x,\ell)\,P^{k'}(\ell,z) \geq P^k(x,y)\,P^{k'}(y,z) > 0. \qquad \square$$

$P(i,j) \geq 0 \quad \forall i,j$

irreducible: A chain is irreducible if it has a unique communicating class.

essential: x is essential if $x \to y \Rightarrow y \to x$ $\forall y$, i.e., x communicates with each state accessible from it.

recurrent / transient: x is recurrent if starting from x, the chain returns to x with probability 1. Otherwise, x is transient.