Math 374

Sandpile model

sandpile graph: $G=(V,E,s)$, directed multigraph, a **sink** vertex $s\in V$

Each $v\in V$ has a (directed) path to s.

$\tilde{V}=V\setminus\{s\}$.

sandpile: $c\in \mathrm{Config}_{\geq 0}(G)=\mathbb{N}\tilde{V}=\{c=\sum_{v\in\tilde{V}} c(v)\,v : c(v)\in\mathbb{N}\}$.

firing rule: If $c(v)\geq \mathrm{outdeg}(v)$, then v is **unstable** and it is **legal** to fire v. The firing rule is like it was with the dollar game:

$$c \xrightarrow{v} c-\mathrm{outdeg}(v)\,v+\sum_{\substack{(v,w)\in E\\ w\neq s}} w.$$

stabilization: (Existence + uniqueness theorem) By repeatedly firing unstable vertices (a **legal firing sequence**) a sandpile c eventually reaches a stable state, i.e., every vertex is stable. The resulting sandpile is called the **stabilization** of c and denoted c°. It is independent of the ordering of legal vertex firings used to reach it.

firing script: Say $c \to \tilde{c}$ via a legal firing sequence $v_1, v_2, \ldots, v_k$. Then $\sigma = \sum v_i \in \mathbb{N}\tilde{V}$ is called <u>the</u> **firing script** for the stabilization of c.

We have $c - \tilde{L}\sigma = \tilde{c}$
(so $\sigma = \tilde{L}^{-1}(c - \tilde{c})$.)

recurrent: A sandpile c is **recurrent** if for all $a \in \mathrm{Config}(G)$ $\exists\, b \in \mathrm{Config}_{\geq 0}(G)$ s.t. $(a+b)^\circ = c$.

sandpile group: The set of recurrent sandpiles with operation $c \circledast c' := (c+c')^\circ$ forms a group called the sandpile group, denoted $S(G)$. We have

ordinary addition

$$S(G) \cong \mathbb{Z}\tilde{V} / \text{image}(\tilde{L})$$

If we define $\text{Jac}(G) = \text{Div}^\circ(G) / \text{image}(L)$, as before but now extended to directed graphs, then there exist a short exact sequence

$$0 \to \mathbb{Z}_{\tau(s)} \to \text{Jac}(G) \to S(G) \to 0$$

where $\tau(s)$ is the s-component of a generator for $\ker L$.

Then $\text{Jac}(G) \cong S(G) \oplus \mathbb{Z}_{\tau(s)}$.

The sandpile group is independent of the choice of sink iff G is Eulerian ($\mathrm{indeg}(v) = \mathrm{outdeg}(v)\ \forall v \in V$). In this case

$$S(G) \simeq \mathrm{Jac}(G)$$

$$c \longmapsto c - \deg(c)s$$

maximal stable configuration: Define $c_{max} := \sum_{v \in \tilde{V}} (\mathrm{outdeg}(v) - 1) v$.

Then:

* a sandpile c is recurrent iff $c = (c_{max} + b)^{\circ}$ for some sandpile b.

* The identity of $S(G)$ is

$$\left(2c_{max} - (2c_{max})^{\circ}\right)^{\circ}$$

☆☆ * (duality) There is an involution

$$\text{recurrents} \longleftrightarrow \text{superstables}$$

$$c \longmapsto c_{max} - c$$

Note: Each equivalence class in $\mathbb{Z}\tilde{V}/\text{image}(\tilde{L})$ has a unique recurrent representative and a unique superstable representative. It is usually <u>not</u> the case that $c = c_{max} - c \mod \tilde{L}$.

burning configuration: There exists a sandpile b called the burning configuration such that c is recurrent iff $(c+b)^\circ = c$. If G is Eulerian, the b is the configuration obtained by firing the sink: $b = \sum_{(s,v)\in E} v$. In that case c is recurrent iff in the stabilization of $c+b$, each vertex fires exactly once.

least-action principle

Let σ be the firing script for $c \to c^\circ$.

Let $\tilde{c} \in \mathrm{Config}(G)$ having no unstable vertices.

Suppose $c \to \tilde{c}$ through a sequence of vertex firings, possibly even illegal firings, with firing script τ, i.e., $c - \tilde{L}\tau = \tilde{c}$ with $\tilde{c}$ stable.

Then $\sigma \le \tau$.

Example