Math 374

HW for this week: L = Laplacian of G, J = matrix of all 1's.

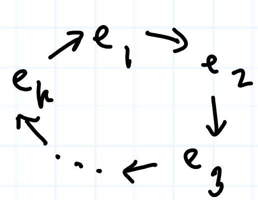
Then # spanning trees of $G = \frac{1}{n^2}\det(L+J)$ where n = # vertices.

Example $G = K_n$. Then $\frac{1}{n^2}\det(L+J) = \frac{1}{n^2}\det\begin{bmatrix} n & & & \\ & n & & \\ & & \ddots & \\ & & & n \end{bmatrix} = n^{n-2}$.

This works for simple graphs. OK directed multigraphs?

Rotor routing G directed multigraph for which every vertex has a path to the "sink" $q \in V$.

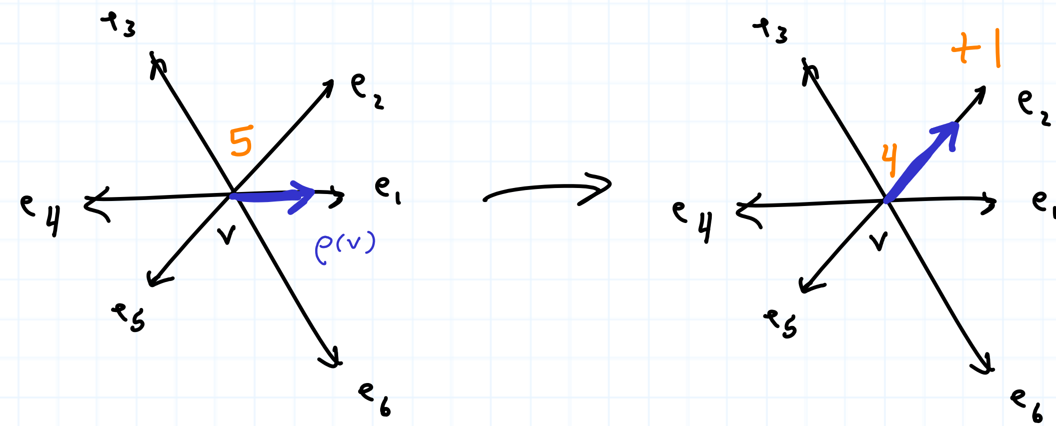
A rotor configuration on G is a mapping $\rho : \tilde{V} \to E$. A rotor state is a pair (c, ρ) where $c \in \mathrm{Config}(G, q)$ and ρ is a rotor configuration. For each $v \in \tilde{V}$, fix a cyclic ordering of the edges with tail v:



If $\rho(v) = e_i$, let $\rho(v)_{next} = e_{i+1}$ (index mod k).

Rotor router operation/update of (c,ρ): Choose $v \in \tilde{V}$ s.t. $c(v) > 0$. Then

call this a vertex firing
$$\left\{ \begin{array}{l} (1)\ \text{Let } e = \rho(v) \\ (2)\ \text{Replace } \rho(v) \text{ by } \rho(v)_{\text{next}}. \\ (3)\ \text{Replace } c \text{ by } c - v + e^{+}. \end{array} \right.$$



Theorem due to Levine + Propp (et al?):

* Starting with (c,ρ) and $c \geq 0$, repeatedly applying rotor-routing operations eventually ends with all grains of sand (dollars) routed to q. The rotor configuration at that point, denoted $c(\rho)$, is independent of the order of vertex-firings.

* If $a, b \in \text{Config}_{\geq 0}(G)$, then

  (i) $(a+b)(\rho) = a(b(\rho))$.

  (ii) $a \equiv b \mod \tilde{L}$ iff $a(\rho) = b(\rho)$.

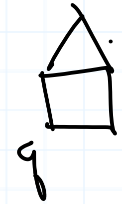
* A rotor configuration ρ is acyclic (which means it forms a directed spanning tree rooted at q) iff for all rotor configurations ρ', $\exists\ c \in \mathrm{Config}_{\geq 0}(G)$ such that $c(\rho') = \rho$. (So ρ is recurrent for the Markov process consisting of choosing a vertex $v \in \tilde{V}$ at random, letting $c = v$, and updating by $\rho' \to c(\rho')$.)

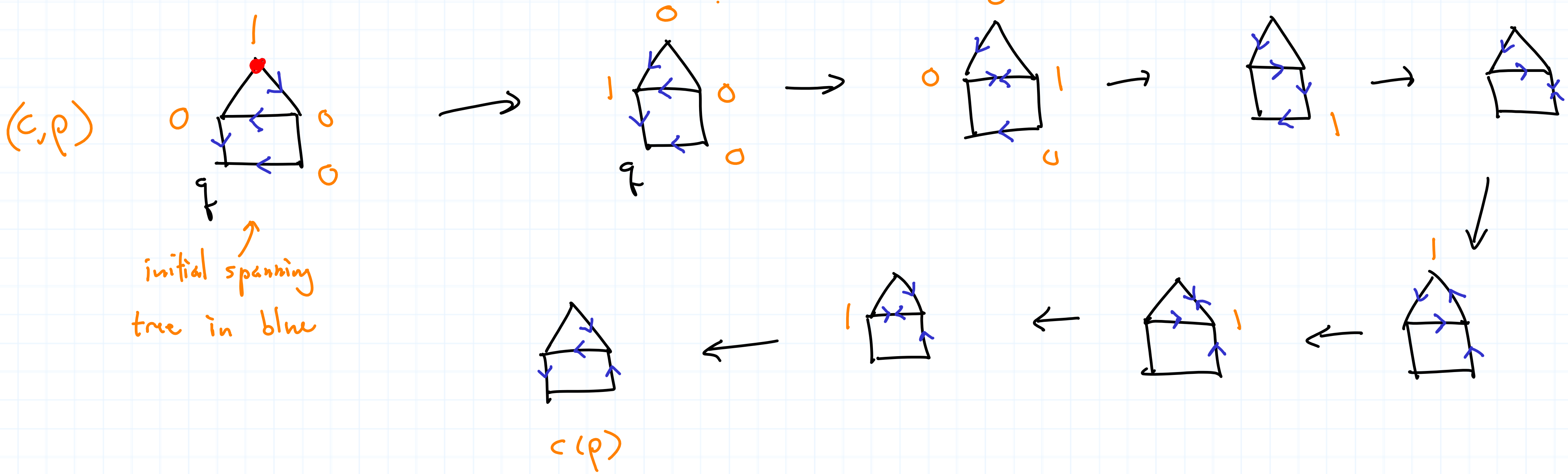
* There is a free and transitive action of $\mathrm{Jac}(G)$ on the set of spanning trees $ST(G,q)$ rooted at q:

$$\mathrm{Jac}(G) \times ST(G,q) \to ST(G,q)$$

$$([D], T) \mapsto c(T)$$

for any choice $c \geq 0$ s.t. $c - \deg(c)q \sim D$.

Example $G =$  with counter-clockwise cyclic ordering of edges at every vertex.



Thm. (Chan, Church, Grochow, 2014) G undirected multigraph with no loops. The rotor-routing action of $\mathrm{Jac}(G)$ on trees is independent of the chosen vertex iff G is planar and w/ cyclic ordering induced by an orientation of the plane (i.e., clockwise at all vertices or counterclockwise at all vertices).

Thm. (Chan, et. al, 2014) The rotor routing action on spanning trees of planar graphs plays nicely with planar duality.