Tree bijections

Math 374

Does what I am about to say generalize to multigraphs?

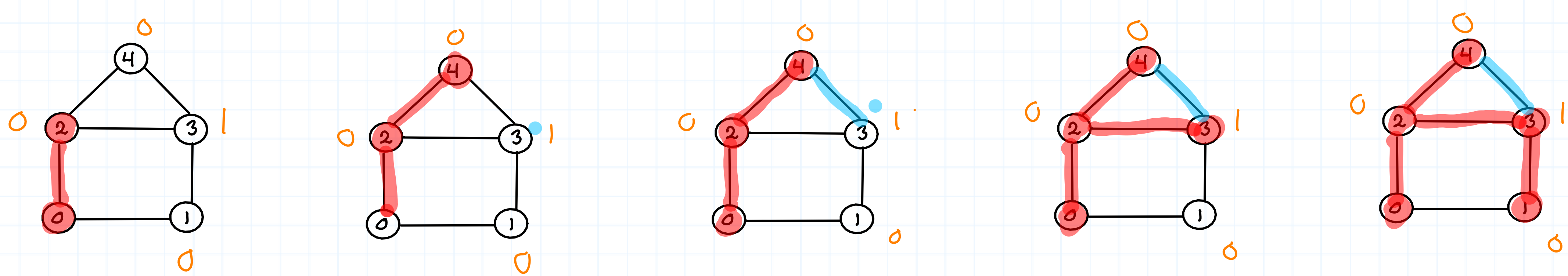
Let $G=(V,E)$ be a simple connected graph with $V=\{0,1,\ldots,n\}$.

Let c be a superstable with respect to vertex 0.

Run Dhar's algorith (with firefighters) using depth-first search:

* Start with 0 = active vertex and ignite 0.
* If i is the active vertex, to decide which edge ij to try to burn next, choose the one with the largest j. If there are no more edges to try, back track to the previously added marked edge. When an edge burns through, make the newly-ignited vertex the active vertex.

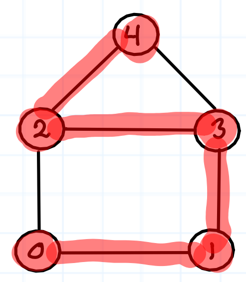
Example

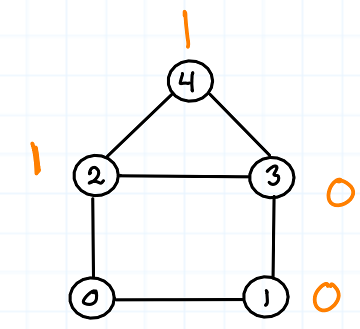


Exercise: everyone tries this at the board using a different superstable. Get a neighbor to check your work.

Inverse: Given a tree, find the superstable it must have come from.

Example





This spanning tree must come from this superstable

So "depth-first search (DFS) - Dhar" gives a bijection

$$\phi : \text{Super}(G) \longrightarrow ST(G)$$

Tree inversions

Let T be a spanning tree. Consider it to be "rooted" at $0 \in V$.
Let i, j be distinct vertices, if i lies on the (unique) path from 0 to j in T, the i is an ancestor of j and j is a descendant of i. If, in addition, $\{i,j\} \in E$, then i is the parent of j and j is a child of i. The pair (i,j) is an inversion of T if

(1) i is an ancestor of j and

(2) $i > j$.

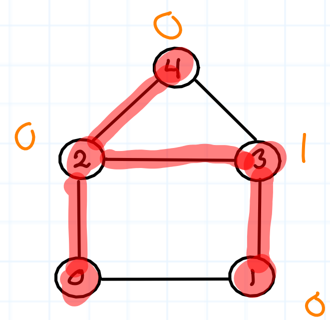
If, in addition (i,j) satisfies

(3) $i \neq 0$ and $\{i', j\} \in E$ where i' is the parent of i,

then (i,j) is a K-inversion.

Exercise for class: count the number of K-inversions for your DFS-Dhar tree.

Example



Inversions: $(2,1)$, $(3,1)$

K-inversions: $(2,1)$ (since $\{0,1\} \in E$)

$(3,1)$ is not a K-inversion since $\{2,0\} \notin E$.

Thm. (D, Kuai, Qiaoyu, 2013) Consider the DFS-Dhar bijection:

$$\phi : \text{Super}(G) \longrightarrow ST(G)$$

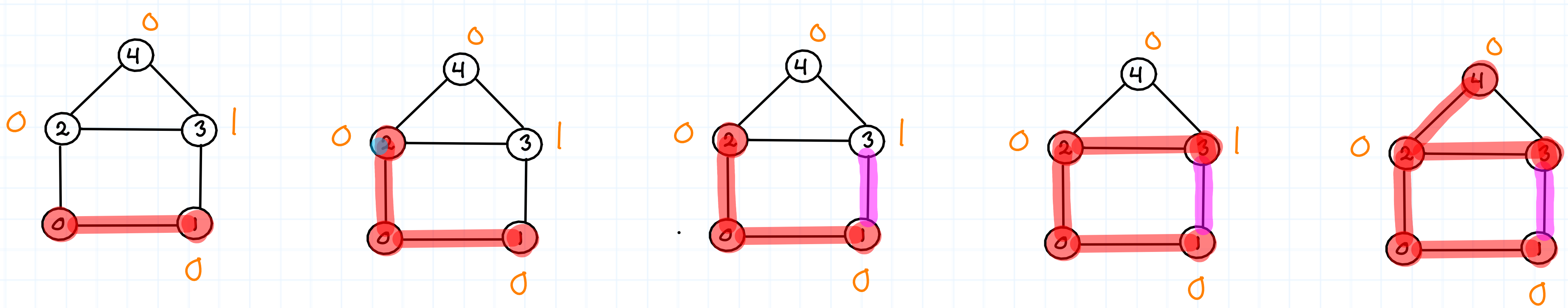
Then, # K-inversions of $\phi(c) = g - \deg(c)$.

Cor. Let $G = K_n$. Then # inversions of $\phi(T) = g - \deg(c)$.

A similar (previous) result by Cori and Le Borgne.

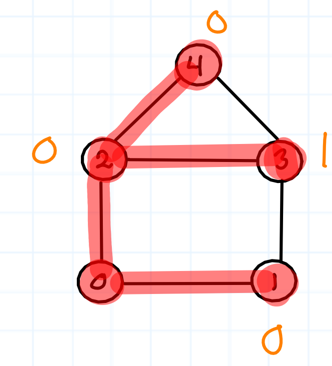
Choose an ordering of the edges, E. Run Dhar, but decide on which edge to try next by choosing the smallest eligible edge.

Example Order the edges lexicographically: 01, 02, 13, 23, 24, 34

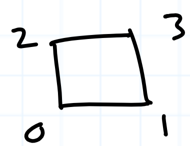


Call an edge $e \in E$ **externally active** (with respect to the chosen ordering for E) for the spanning tree T if $e \notin T$ and e is the largest edge in the unique cycle of $T \cup \{e\}$.

In our example, above, the tree is



T

The edge $\{1,3\}$ is not externally active since it is not the largest edge in the cycle 

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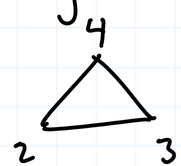
The edge, $\{3,4\}$ is externally active since it is the largest edge in 

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Thm. (Cori + Le Borgne, 2001). Consider the "edge-ordered-Dhar" bijection

$$\Psi : \mathrm{Super}(G) \longrightarrow ST(G)$$

Then # externally active edges of $\Psi(c) = g - \deg c$. There is a bijection between "untried" edges in the application of Dhar and externally active edges.

Exercise for class: Check this result for your tree.