Math 374

## Matrix-tree theorem

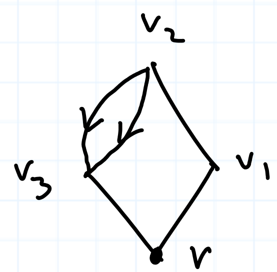
Let $G=(V,E)$ be a directed multigraph. A directed spanning tree with root $v \in V$ is an acyclic subgraph $T$ of $G$ containing all of the vertices such that each vertex $u \neq v$ is the tail of exactly one edge of $T$ and $v$ is not the tail of any edge of $T$.

The matrix-tree theorem says:

$$\#\text{ spanning trees of } G = \det \tilde{L}$$

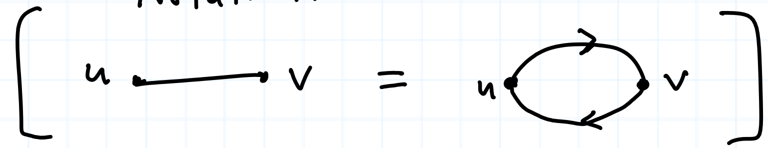
where $\tilde{L}$ is the reduced Laplacian of $G$ with respect to $v$.

## Example

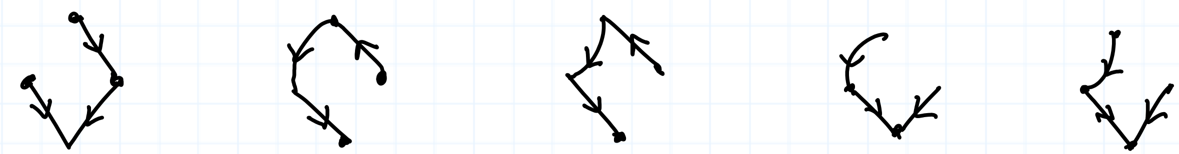



$G$

Notation:




Spanning trees rooted at $v$:



$$\det \tilde{L} = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \det \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = 5.$$

We'll prove a somewhat more general version. Let $G = (V, E)$ directed graph on vertex set $V = \{1, \dots, n\}$. Let $\{x_{ij} : i, j \in \{1, \dots, n\}\}$ be indeterminates and set $x_{ij} = 0$ if $ij \notin E$. We interpret the $x_{ij}$ as edge weights or multiplicities. (Setting the $x_{ij}$ equal to natural numbers models directed multigraphs.)

If $T$ is a directed spanning tree of $G$ into $v \in V$, define the weight of $T$ to be the product of the weights of its edges. Define the number of weighted directed spanning trees of $G$, denoted, $\tau(G,v)$, to be the sum of the weights of all directed spanning tre s of $G$ into $v$.

Define the matrix $\Delta = \Delta(G)$, by

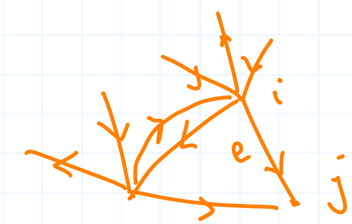
$$\Delta_{ij} = \begin{cases} \sum_{k \neq i} x_{ik} & \text{if } i = j \\ -x_{ij} & \text{if } i \neq j \end{cases}.$$

$\Delta$ is a generic version of the Laplacian matrix (or its transpose, depending on conventions).

Define the reduced-$\Delta$ by $\tilde{\Delta} = \tilde{\Delta}(G,v)$, with respect to $v \in V$ by removing the row and column of $\Delta$ indexed by $v$.

Given $e \in E$, let $G \setminus e$ be the graph obtained from $G$ by deleting $e$ (essentially, setting $x_e = 0$). Let $G/e$ be the graph obtained from $G$ by contracting $e$. (If $e = ij$, identify vertices $i$ an $j$. We can imagine that $e$ becomes a loop or is deleted. It will not matter for our purposes here.)

**Lemma** For each edge $e = ij$,




$$\tau(G,j) = \tau(G \setminus e, j) + x_{ij}\,\tau(G/e, j).$$

Pf/ There are two kinds of spanning trees of $G$ into $j$: those that do not contain $e$, and those that do. $\square$

Matrix - tree

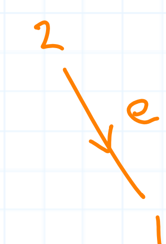
$$\tau(G,v) = \det \tilde{\Delta}(G,v).$$

Pf/ We prove this by induction on the number of edges of $G$. Let $e = ij \in E$. The result follows from the lemma, above, if we can show

$$\det \tilde{\Delta}(G,j) = \det \tilde{\Delta}(G \setminus e, j) + x_{ij} \det \tilde{\Delta}(G/e, j).$$

(The base case, in which $G$ has a single edge, is trivial.)

First, $\tilde{\Delta}(G \setminus e, j)$ is obtained from $\Delta(G,j)$ by subtracting the $i^{\text{th}}$ standard basis vector from row $i$. The result then follows by multilinearity of the determinant. In detail, suppose $i = 2$ and $j = 1$.




Let the rows of $\tilde{\Delta}(G,1)$ be $r_2, \dots, r_n$. Then the rows of $\tilde{\Delta}(G \setminus e, 1)$ are $r_2 - e_2, r_3, \dots, r_n$ where $e_2 = (0,1,0,\dots,0)$. Therefore,

$$\det \tilde{\Delta}(G \setminus e, j) = \det(r_2, \ldots, r_n) - \det(e_2, r_3, \ldots, r_n)$$

$$= \det(r_2, \ldots, r_n) - \det(r_3, \ldots, r_n)$$

$$= \det \tilde{\Delta}(G, 1) - \det \tilde{\Delta}(G/e, \tilde{1}).$$

(where $\tilde{1}$ is the vertex obtained by identifying vertices 1 and 2 in $G$).

$\square$