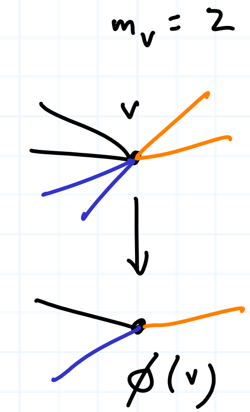
Math 374

edges may be collapsed

Recall: $\phi : G \to G'$ $\quad (\phi : V \cup E \to V' \cup E')$



harmonic at $v \in V$ if the following number is independent of the edge e' incident with $\phi(v)$:

$$m_v =: \#\text{ pre-images of } e' \text{ incident to } v$$

$$= \#\{ vw \in E \text{ s.t. } \phi(v)\phi(w) = e' \}.$$

multiplicity of v

Then ϕ is harmonic if it's harmonic $\forall v \in V$.

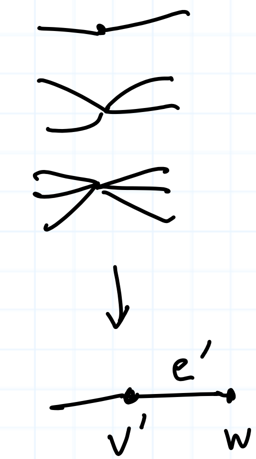
Prop. If $\phi : G \to G'$ is a nonconstant harmonic mapping, then

$$\deg \phi := \# \phi^{-1}(e')$$

is independent of $e' \in E'$ and for each $v' \in V'$,

$$\deg \phi = \sum_{v \in \phi^{-1}(v') \cap V} m_v .$$

Pf/ Let $e' = v'w'$ Then



$$\# \phi^{-1}(e') = \# \{ vw \in E : \phi(vw) = e' \}$$

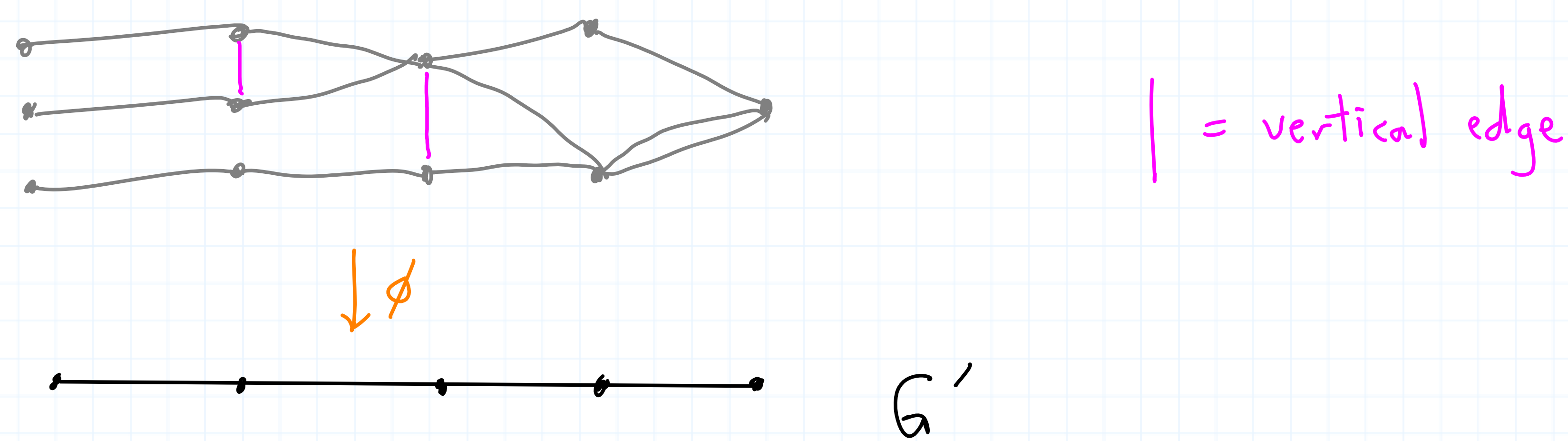
$$= \sum_{v \in \phi^{-1}(v') \cap V} \# \{ e = vw \in E : \phi(e) = e' \}$$

$$= \sum_{v \in \phi^{-1}(v') \cap V} m_v$$

Since ϕ is harmonic, this shows this number is the same for all edges adjacent to v'. Then, since G' is constant, the result follows. $\square$

This proposition leads to a way of thinking of non-constant harmonic mappings. If $\phi : G \to G'$ is a non-constant harmonic mapping of degree d, what can G look like? To construct G start with d copies of G' stacked above G'. Above each each edge

e' of G', there are now d copies of e', and above each $v' \in V'$, there are d copies of v'. Then, for each $v' \in V'$, we are allowed to identify the vertices in subsets of the d copies of v' sitting above $v' \in V'$. Having done this, we are allowed to add vertical edges. We also want to make sure G is connected.



Prop. Let $\phi : G \to G'$ be a non-constant harmonic mapping.

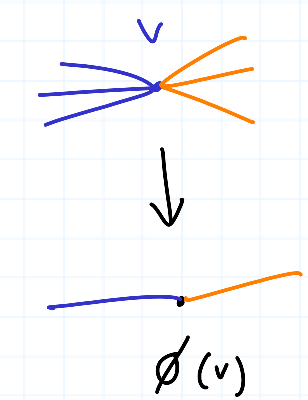
Then $\phi_* : \mathrm{Div}(G) \to \mathrm{Div}(G')$

$$D = \sum_{v \in V} a_v v \longmapsto \sum_{v \in V} a_v \phi(v)$$

induces surjective homomorphisms

$$\phi_* : \mathrm{Pic}(G) \to \mathrm{Pic}(G') \quad \text{and} \quad \phi_* : \mathrm{Jac}(G) \to \mathrm{Jac}(G').$$

Pf/ Vertex borrowings generate $\mathrm{Prin}(G)$. Let $v \in V$. Then



$$\begin{aligned}
\phi_*\Big(\underbrace{\sum_{vw \in E} (v-w)}_{\text{borrowing at } v} \Big) &= \sum_{vw \in E} (\phi(v) - \phi(w)) \\
&= \sum_{v'\phi(v) \in E'} m_v (\phi(v) - v') \\
&= m_v \sum_{v'\phi(v) \in E'} \underbrace{(\phi(v) - v')}_{\text{firing } \phi(v)} .
\end{aligned}$$

Riemann-Hurwitz theorem for graphs. (Baker + Norine, 2009)

Let $\phi : G \to G'$ be a nonconstant harmonic morphism.

Let g = genus of G and g' = genus G'. Then:

$$2g-2 = \deg(\phi)(2g'-2) + \sum_{v \in V} \left[2(m_v - 1) + \mathrm{vert}(v)\right]$$

where $\mathrm{vert}(v)$ = number of vertical edges incident to v.

In fact, if we define $\phi^* : \mathrm{Div}(G') \to \mathrm{Div}(G)$ by

$\phi(D') = \sum_{v \in V} m_v D'(\phi(v))v$, there are induce injections

$$\phi^* : \mathrm{Pic}(G') \hookrightarrow \mathrm{Pic}(G) \quad \text{and} \quad \phi^* : \mathrm{Jac}(G') \hookrightarrow \mathrm{Jac}(G).$$

Let K and K' be the canonical divisors on G and G', and define the ramification divisor for ϕ by

$$R := 2\sum_{v \in V} (m_v - 1)\, v + \sum_{v \in V} \operatorname{vert}(v)\, v.$$

Then $K \overset{\star}{=} \phi^* K_{G'} + R$

Exercise: $\deg \phi^*(D') = \deg(\phi) \deg(D') \quad \forall D' \in \operatorname{Div}(G')$

So the Riemann-Hurwitz formula follows from taking the degree of both sides of $\star$.