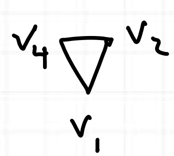
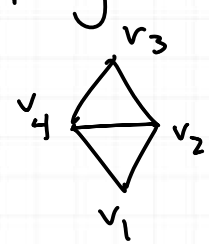
Math 374

Harmonic morphisms.

What are the appropriate morphisms between graphs for our subject? Given an arbitrary mapping of graphs $\phi : G \to G'$, there is a natural "pushforward" mapping for divisors, $\phi_* : \mathrm{Div}(G) \to \mathrm{Div}(G')$ given by $\phi_*(D)(v') = \sum_{v \in \phi^{-1}(v')} D(v) \quad \forall v' \in V(G')$. For a "nice" mapping, we might hope that ϕ_* induces a mapping $\phi_* : \mathrm{Jac}(G) \to \mathrm{Jac}(G')$. This does not work in general, even in the case of an inclusion mapping.

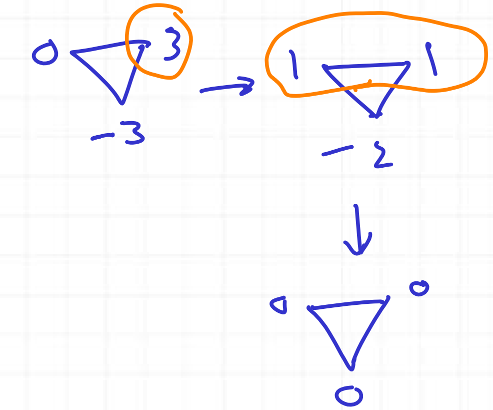
Example $\phi : G =$ 

$\hookrightarrow$ 

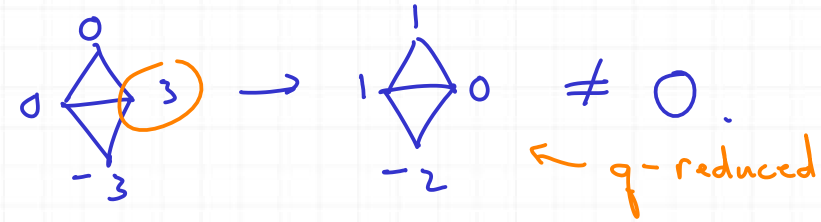
$= G'$

$$v_i \mapsto v_i$$

Consider $D = v_2 - v_1$. Then $3(v_2 - v_1) = 0$ in $\mathrm{Jac}(G)$.



What about $3\phi_*(D)$?



Problem!

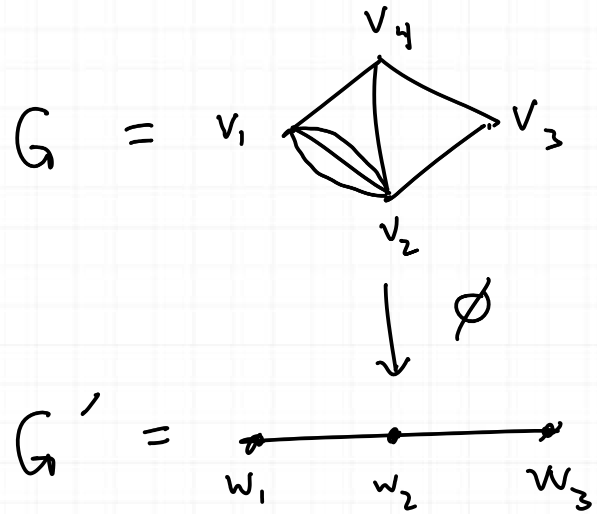
Define a graph morphism between $G = (V, E)$ and $G' = (V', E')$ to be a set mapping $\phi : V \cup E \to V' \cup E'$ such that

$$\phi(v) \in V' \quad \text{if } v \in V$$

$$\phi(uv) = \begin{cases} \phi(u)\phi(v) \in E' & \text{if } \phi(u) \neq \phi(v) \\ \phi(v) \in V' & \text{if } \phi(u) = \phi(v). \end{cases}$$

So our morphism are ordinary mappings of graphs except that edges are allowed to "collapse".

Example

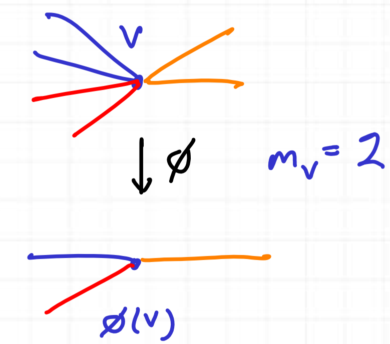


Let $\phi(v_1) = w_1$, $\phi(v_2) = \phi(v_4) = w_2$, and $\phi(v_3) = w_3$.

Def. A graph morphism $\phi : G \to G'$ is **harmonic** at $v \in V$ if the following quantity is independent of the edge $e' \in E'$ incident to $\phi(v) \in V'$:

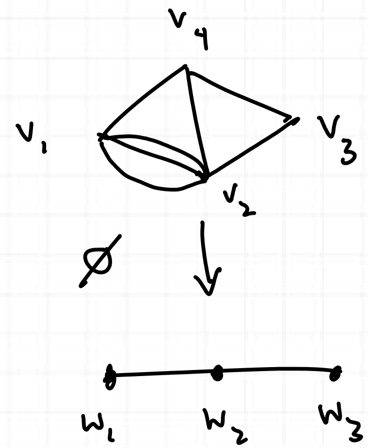
$$m_v = \#\text{ pre-images of } e' \text{ incident with } v$$

$$= \#\{ vw \in E : \phi(vw) = e' \}.$$



The number m_v is the **horizontal multiplicity** of v. The morphism ϕ is **harmonic** if it is harmonic at each $v \in V$

Examples



$\phi(v_1) = w_1$, $\phi(v_2) = \phi(v_4) = w_2$, $\phi(v_3) = w_3$.

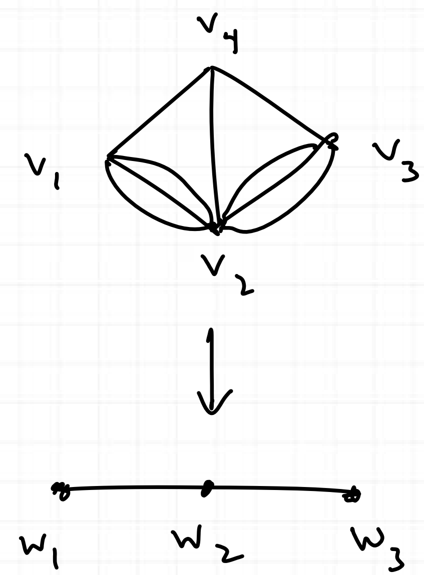
This ϕ is harmonic at v_1 and at v_3 since there is only one edge incident to w_1 and only one edge incident to w_3.

What about v_4? There are two edges incident with $\phi(v_4) = w_2$: $e_1 = w_1 w_2$ and $e_2 = w_2 w_3$. Each of these has exactly one edge in its preimage incident with v_4, namely $v_1 v_4$ and $v_3 v_4$, respectively.

What about v_2? There are 3 edges in the preimage of $w_1 w_2$ incident with v_2 and only 1 edge in the preimage of $w_2 w_3$ incident with v_2. So ϕ is not harmonic at v_2.

The following mapping is harmonic:

$\phi(v_1) = w_1, \ \phi(v_2) = \phi(v_4) = w_2, \ \phi(v_3) = w_3.$



Exercise. Create more examples of harmonic morphisms.

Facts (To be proved later) Let $\phi : G \to G'$ be a non constant harmonic morphism. Then

* ϕ is surjective on vertices and on edges

* The number of preimages of an edge of G' is the same for for all edges of G'. This number is called the degree of ϕ [☆ Think about factorization of primes in a field extension. Mention Galois theory. ☆]

* The pushforward mapping on divisors induces surjective mappings of groups

$$\phi_* : \mathrm{Pic}(G) \twoheadrightarrow \mathrm{Pic}(G') \quad \text{and} \quad \phi_* : \mathrm{Jac}(G) \twoheadrightarrow \mathrm{Jac}(G')$$

* For all $D \in \mathrm{Div}(G)$, we have $r(\phi_*(D)) \geq r(D)$.
(The "economy becomes more robust".)

* Define a pullback mapping $\phi^*: \mathrm{Div}(G') \to \mathrm{Div}(G)$ by $\phi^*(D')(v) = m_v\, D'(\phi(v))$. Then there are induced injective mappings of groups

$$\phi^*: \mathrm{Pic}(G') \hookrightarrow \mathrm{Pic}(G) \quad \text{and} \quad \phi^*: \mathrm{Jac}(G') \hookrightarrow \mathrm{Jac}(G).$$

Questions:

* Are there categorical products? Coproducts?
* Is there a way of thinking of the set of harmonic mappings $G \to G'$ as the vertices of a graph in a meaningful way?