Recall:

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* $r(D) = \begin{cases} -1 & \text{if } \overbrace{D \text{ unwinnable}}^{|D| = \emptyset} \\ \max\{k : \underbrace{D-E}_{|D-E| \neq \emptyset} \text{ winnable } \forall E \in \mathrm{Div}^k_{\geq 0}(G)\} \end{cases}$

* D max'l unwinnable $\iff D \sim c - q$ w/ c max'l superstable
  $\iff D = D(\mathcal{O})$ for some acyclic orientation $\mathcal{O}$ with unique source

* If $\mathcal{O}$ is <u>any</u> acyclic orientation, the $D(\mathcal{O})$ is max'l unwinnable.

* Canonical divisor : $K := D(\mathcal{O}) + D(\mathcal{O}^{rev}) = \sum_{v \in V} (\deg_G(v) - 2)v$

$$\deg K = 2g - 2$$

Theorem (Riemann-Roch theorem for graphs; Baker + Norine, 2007)

$$r(D) - r(K-D) = \deg D + 1 - g$$

Pf/ As in book.

Cor. N max'l unwinnable iff $K-N$ maximal unwinnable.

Pf/ As in book. $\square$

Exercise. $r(K) = g-1$.

Cor. (Clifford's theorem.) Suppose $r(D) \geq 0$ and $r(K-D) \geq 0$. Then

$$r(D) \leq \frac{1}{2} \deg D$$

Pf/ As in book: $r(D) + r(K-D) \leq r(K) = g-1$ + RR. $\square$

Cor. (1) $\deg D < 0 \Rightarrow r(D) = -1$

(2) $0 \leq \deg D \leq 2g-2 \Rightarrow r(D) \leq \frac{1}{2} \deg D$

(3) $\deg D > 2g-2 \Rightarrow r(D) = \deg D - g$.

Pf/ As in book. $\square$

Extra time: talk about tropical curves + 2-d grid sandpile paper that was posted on the arXiv yesterday.