# Math 374

The rank of $D \in \mathrm{Div}(G)$ is

$$r(D) = \begin{cases} -1 & \text{if } |D| = \emptyset \\ \max\{k : |D-E| \neq \emptyset \ \forall E \geq 0, \deg E = k\} \end{cases}$$

Roughly: How many dollars can we take away from D and still have a game that is winnable?

Example $G = K_5$ vertices $v_1, \dots, v_5$.

Consider $D_k = \underbrace{v_3 + 3v_4}_{c \text{ superstable}} + k v_5 = (0, 0, 1, 3, k)$

What is $r(D_k)$? If $k \leq -1$, the $D_k$ is unwinnable.

So in that case $r(D_k) = -1$. From Sage:

| $k$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|
| $r(D_k)$ | $-1$ | $0$ | $0$ | $0$ | $1$ | $2$ | $3$ |

Why is $r(D_0) = 0$? $D_0 = (0,0,1,3,0)$, which is winnable.

However, $D_0 - v_5 = (0,0,1,3,-1)$ is unwinnable (since it has the form $c - v_5$ where $c$ is superstable).

Why is $r(D_1) = 0$? $D_1 = (0,0,1,3,1)$. Consider $D_1 - v_1 = (-1,0,1,3,1)$. Taking $q = v_1$ and $c = v_3 + 3v_4 + v_5$, we have $D_1 - v_1 = c - q$ where $c$ is superstable (Recall: On $K_n$, the superstables are all component-wise permutations of configurations

$0 \le c \le (0,1,2,\ldots,n-1)$. We have $(0,1,1,3) \le (0,1,2,3)$, hence, $(0,1,3,1)$ is superstable.)

This reasoning also explains why $r(D_2) = 0$.

$D_2 = (0,0,1,3,2)$. Then $D_2 - v_1 = \text{superstable} - v_1 \Rightarrow D_2 - v_1$ unwinnable.

Why is $r(D_3) = 1$? $D_3 = (0,0,1,3,3)$..

Recall: If $\deg D \ge g$, then $D$ is winnable. In the case of $K_n$, we have $g = \binom{5}{2} - 5 + 1 = 6$. We have $\deg D_3 = 7$. Hence, $\deg(D_3 - v) = 6 \Rightarrow D_3 - v$ is winnable for all $v \in V \Rightarrow r(D_3) \ge 1$.

**Prop.** $r(D) \le r(D+v) \le r(D)+1$ $\forall D \in \mathrm{Div}(G)$ for arbitrary $G$.

*Pf/* First suppose $r(D) = -1$. Then, clearly, $r(D) \le r(D+v)$. Further, if $r(D+v) \ge 1$, then $(D+v) - v = D$ would be winnable, i.e., $r(D) \ge 0$. Hence, $r(D+v) \le 0 = r(D)+1$.

Now suppose $r(D) = k \ge 0$. Let $E \ge 0$ with $\deg E \le k$. Then

$$D - E \sim F \ge 0 \implies (D+v) - E \sim F + v \ge 0.$$

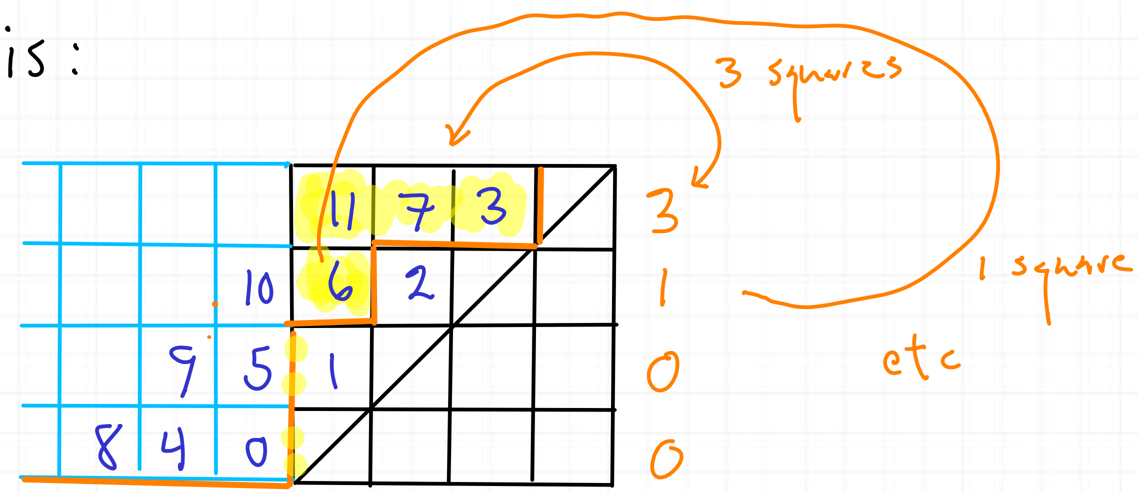
Therefore, $r(D+v) \ge k$. Also, $\exists\, E' \ge 0$ with $\deg E' = k+1$ s.t. $D - E'$ is unwinnable. Consider $E'' := E' + v$. Then $\deg E'' = k+2$, and

$$(D+v) - E'' = D - E'$$

is unwinnable. Thus, $r(D+v) \le k+1$. $\square$

# Cori - Le Borgne

There is an algorithm for computing the rank of a divisor on $K_n$. See the explanation of Figure 7 in Chapter 5 of our text. Applying the idea to our divisor $D_k = (0,0,1,3,k)$ on $K_n$, the relevant picture is:




To compute $r(D_k)$, look up the number $k$, in blue, in the diagram. Count the number of blue numbers $\leq k$ that are not "under the staircase", and subtract 1. For example, $r(D_3) = 1$ since only 3 and 0 are above the staircase. For $k \geq 3$, we see $r(D_k) = k-2$.

* See Sage worksheet for visualizing the complete linear system of a divisor.