Math 374

Thm 4.6 Fix $q \in V$. There is a bijection:

$$\mathfrak{c} : \underbrace{\begin{array}{c}\text{acyclic orientations}\\ \text{with unique source } q\end{array}}_{}{}^{\!\!\!\!\!\!AO(q)} \longrightarrow \text{max'l superstables}$$

$$\mathcal{O} \longmapsto \mathfrak{c}(\mathcal{O}) = \sum_{v \in \tilde{V}} \left(\mathrm{indeg}_{\mathcal{O}}(v) - 1\right) v.$$

Pf/ (continued from last time) We have seen that $\forall \mathcal{O} \in AO(q)$ that $c(\mathcal{O})$ is superstable. [Order the vertices using any linear ordering refining the partial ordering determined by $\mathcal{O}$. Then the vertices may be removed in this order when performing Dhar's algorithm.

Next, we introduce an algorithm producing a mapping:

$$\alpha : \text{superstables} \to AO(q)$$

[If c superstable, run Dhar and when S is fired and vertex v is about to be removed from S, direct all edges $\{w,v\} \in E$ with $w \notin S$ towards v. In the end, we get $\mathcal{O} \in AO(q)$ with $c(v) < \mathrm{indeg}_{\mathcal{O}}(v)$. Therefore,

$$c \leq c(\alpha(c)). \qquad (\star)$$

If c is a max'l superstable, then $c = c(\alpha(c))$.

Now consider $\alpha(c(\mathcal{O}))$ for some $\mathcal{O} \in AO(q)$. Let $c := c(\mathcal{O})$.
By $(\star)$, we have

$$\mathrm{indeg}_{\mathcal{O}}(v) \leq \mathrm{indeg}_{\alpha(c)}(v) \qquad \forall v \in V.$$

Therefore,

$$\# E = \sum_{v \in V} \operatorname{indeg}_{\mathcal{O}}(v) \leq \sum_{v \in V} \operatorname{indeg}_{\alpha(c)}(v) = \# E,$$

$$\Rightarrow \quad \operatorname{indeg}_{\mathcal{O}}(v) = \operatorname{indeg}_{\alpha(c)}(v) \qquad \forall v \in V$$

$$\Rightarrow \quad c = \mathfrak{c}(\alpha(c)).$$

So α is the inverse of $\mathfrak{c}$ restricted to maximal superstables. $\square$

Cor. Let $g := \#E - \#V + 1$ the genus or cyle rank of G.

(1) A superstable c is max'l iff $\deg(c) = g$.

(2) D is a maximal unwinnable iff the q-reduced form of D is $c - q$ where c is a maximal superstable.

(3) The correspondence $\mathcal{O} \to D(\mathcal{O})$ is a bijection between acyclic orientations with unique source q and q-reduced maximal unwinnable divisors.

(4) If D is a maximal unwinnable divisor, then $\deg D = g-1$.
So if $\deg D \geq g$, then D is winnable.

Pf/ (1) The maximal superstables have the form $c(\mathcal{O})$ for some a.o w/ ! source $\mathcal{O}$. Then $\deg c(\mathcal{O}) = \sum_{v \in \tilde{V}} (\mathrm{indeg}_{\mathcal{O}}(v) - 1) = \sum_{v \in \tilde{V}} \mathrm{indeg}_{\mathcal{O}}(v) - \sum_{v \in \tilde{V}} 1$

$= \#E - \#V + 1$.

(2) ($\Longrightarrow$) Straightforward.

($\Longleftarrow$) Let $D = c - q$ w/ c a max'l superstable. Then D is unwinnable. To show it's max'l unwinnable, take $v \in V$. If $v = q$, then $D + v = c$ is winnable (already won!). If $v \neq q$,

$D + v = (c+v) + q$ with $c+v \in \text{Config}(G, q)$. But we've part (1) says $\deg(c+v) = g+1$ and the stabilization of $c+v$ has degree $\leq g$.

Hence, computing the q-reduced form of $D+v$ by stabilizing $c+v$ sends at least one dollar to q, winning the dollar game.

(3) & (4) follow easily from (1) and (2). $\square$