Math 374

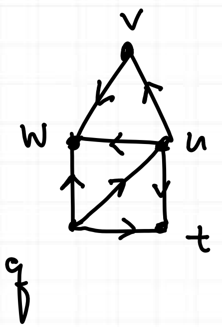
Thm 4.6 Fix $q \in V$. There is a bijection:

$$\mathfrak{c} : \begin{array}{c}\text{acyclic orientations} \\ \text{with unique source } q\end{array} \longrightarrow \text{max'l superstables}$$

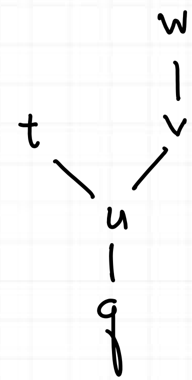
$$\mathcal{O} \longmapsto \mathfrak{c}(\mathcal{O}) = \sum_{v \in \tilde{V}} (\operatorname{indeg}_{\mathcal{O}}(v) - 1)\, v .$$

Pf/ Let $\mathcal{O}$ be an acyclic orientation w/ unique source q, and let $c = \mathfrak{c}(\mathcal{O})$. We first show c is superstable. Order the vertices so that if $(u,v) \in \mathcal{O}$, then $u < v$. There may be several choices.

Example



θ



partial ordering on vertices determined by θ

$\rightsquigarrow \quad q < u < v < w < t$

choose any refinement to a linear order (t can be placed anywhere after u)

Now run Dhar's algorithm. We claim that the non-q vertices may be removed according to our order. Let u be the vertex just after q in our ordering. The edges directed into u are exactly (q,u) for each edge of the form $\{q,u\}$. So when Dhar begins with $S = \tilde{V}$, we have

$$c(u) = \mathrm{indeg}_\theta(u) - 1 = \mathrm{outdeg}_S(u) - 1 < \mathrm{outdeg}_S(u).$$

So u may be removed. By induction assume Dhar has run for a while and exactly the vertices preceding some $v \in \tilde{V}$ have been removed from S. What is $\operatorname{outdeg}_S(v)$? Suppose $\{w,v\} \in E$ with $w \notin S$. Since w precedes v, we have $(v,w) \notin \mathcal{O}$. Hence, (w,v) is added to $\mathcal{O}$. every edge is directed in $\mathcal{O}$, we must have $(w,v) \in \mathcal{O}$. Conversely, if $(t,v) \in \mathcal{O}$, we must have $t \notin S$. Therefore, $\operatorname{outdeg}_S(v) = \operatorname{indeg}_{\mathcal{O}}(v)$ $= c(v)+1$. So v may be removed.

We now define a mapping

$$a : \text{superstables} \longrightarrow \text{acyclic orientations w/ ! source } q$$

via a Dhar-like algorithm. Let c be superstable. Start with $\mathcal{O} = \emptyset$. Run Dhar, and whenever a set S is fired and it is determined that $v \in S$ should be removed from S, i.e. $c(v) < \operatorname{outdeg}_S(v)$ add $(u,v) \in \mathcal{O}$ for each $\{u,v\} \in E$ s.t. $w \notin S$. We have $c(v) \overset{*}{<} \operatorname{outdeg}_S(v) = \operatorname{indeg}_{\mathcal{O}}(v)$. In the end we get an acyclic orientation $a(c) := \mathcal{O}$ with unique source q such that $c(v) \leq c(a(c))$ (by ☆).

We have seen that $c(a(v))$ is superstable. So if c is max'l, we must have $c = c(a(c))$. Hence, on max'l superstables, a is injective.

Now consider $a(c(\theta))$ for some acy. orient. w/ ! source q.

Let $c := c(\theta)$. Then $c \leq c(a(c)) \Rightarrow \mathrm{indeg}_{\theta}(v) \leq \mathrm{indeg}_{a(c)}(v)$ $\forall v \in \tilde{V}$.

$$\sum_{v \in \tilde{V}} \mathrm{indeg}_{\theta}(v) = \sum_{v \in \tilde{V}} \mathrm{indeg}_{a(c)}(v) = |E|.$$

So $\mathrm{indeg}_{\theta}(v) = \mathrm{indeg}_{a(c)}(v)$ $\forall v \in V$, and since acyclic orientations are determined by their indegree sequences, $\theta = a(c) = a(c(\theta))$.

So a and c restricted to maximal superstables are inverses. $\square$