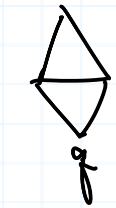
Acyclic Orientations

* $D \in \mathrm{Div}(G)$ is maximal unwinnable if whenever $D' \in \mathrm{Div}(G)$ is unwinnable and $D \le D'$, then $D' = D$.

* $c \in \mathrm{Config}(G, q)$ is a maximal superstable if whenever $c' \in \mathrm{Config}(G, q)$ is superstable and $c \le c'$, then $c' = c$.

* D max'l unwinnable iff D unwinnable and $D + v$ winnable $\forall\, v \in V$
  c max'l superstable iff c superstable and $c + v$ not superstable $\forall\, v \in \tilde{V}$.

Questions. Is each unwinnable divisor is dominated by some max'l unwinnable?
Is each superstable is dominated by some max'l superstable?

Example What are the maximal superstables of  ?
What are its maximal unwinnable divisors?

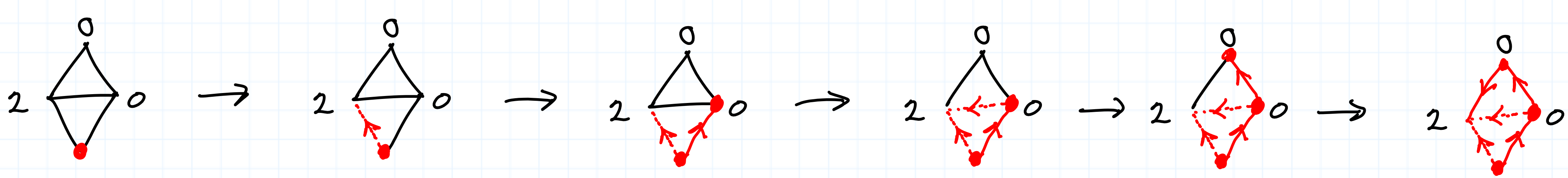
Question: If D is maximal unwinnable and $D \sim D'$ is D' maximal unwinnable?

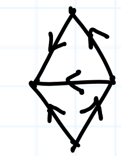
$$\begin{array}{ccc} D & \le & D' \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \tilde{D} & \le & \tilde{D}' \end{array}$$

Fact. D is max'l unwinnable iff its q-reduced form is $c - q$ where c is a maximal superstable.

* Only "$\Rightarrow$" is straightforward to show.

What happens if we run Dhar's algorithm on a maximal superstable?



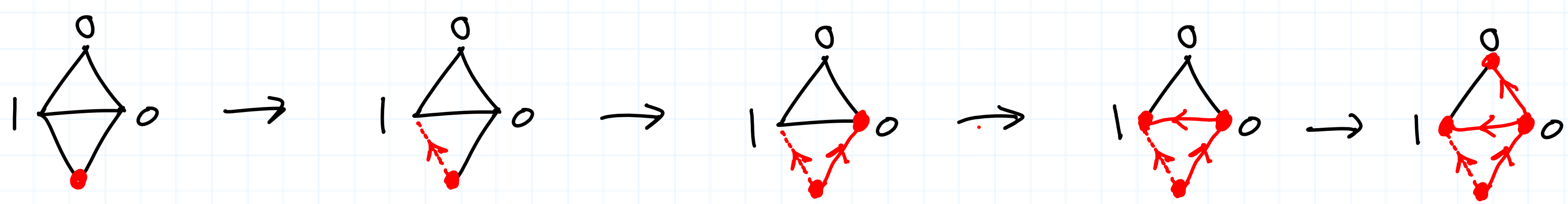
Result: an acyclic orientation of the graph: 

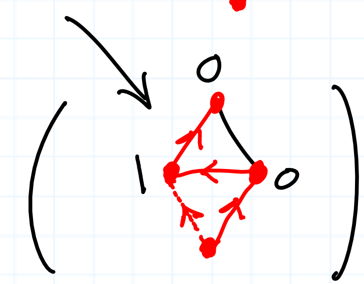
Questions: Why is it acyclic?

Why does it have a unique source?

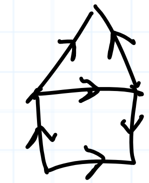
Why is every edge oriented?

What if we run Dhar's on a non-maximal superstable?



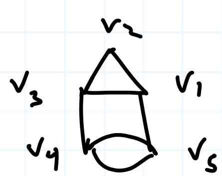


Note that in the end, not every edge is oriented.

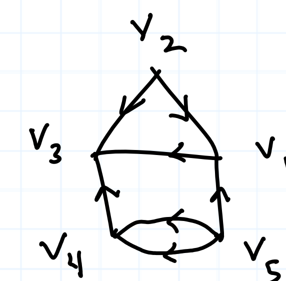
In reverse: Which superstable is associated to  ?

* Define $\text{indeg}_\theta(v)$.

* **Lemma 4.2** An acyclic orientation is characterized by its indegree sequence.

Example Which acyclic orientation of 

has indegree sequence $(2,0,3,2,0)$?
$\quad\ v_1\ v_2\ v_3\ v_4\ v_5$

Solution 

Proof of lemma 4.2/ As in text.

Why does every acyclic orientation have a sink?

* Divisor + config. associated with an orientation θ:

$$D(\theta) := \sum_{v\in V} (\text{indeg}_\theta(v)-1)v \quad , \quad c(\theta) = \sum_{v\in \tilde{V}} (\text{indeg}(v)-1)\,\theta.$$

Thm 4.6 Fix $q \in V$. There is a bijection:

$$\mathfrak{c} : \begin{array}{c}\text{acyclic orientations} \\ \text{with unique source } q\end{array} \longrightarrow \text{max'l superstables}$$

$$\mathcal{O} \longmapsto \mathfrak{c}(\mathcal{O}) = \sum_{v \in \widetilde{V}} \left(\operatorname{indeg}_{\mathcal{O}}(v) - 1\right) v .$$